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A Splitting Algorithm for Coupled System of Primal–Dual Monotone Inclusions

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Abstract We propose a splitting algorithm for solving a coupled system of primal-dual monotone inclusions in real Hilbert spaces. The proposed algorithm has a structure identical to that of the forward-backward algorithm with variable metric. The operators involved in the problem formulation are used separately in the sense that single-valued operators are used individually and approximately in the forward steps and multi-valued operators are used individually via their generalization resolvent in the backward steps. The weak convergence of the algorithm proposed is proved. Applications to coupled system of monotone inclusions in duality and minimization problems, and multi-dictionary signal representation are demonstrated.

Keywords coupled system · monotone inclusion · monotone operator · operator splitting · cocoercivity · forward-backward algorithm · composite operator · duality · primal-dual algorithm

Mathematics Subject Classification (2000) 47H05 · 49M29 · 49M27 · 90C25

1 Introduction

Various problems in applied mathematics such as evolution inclusions [1], partial differential equations [2–4], mechanics [5], variational inequalities [6, 7], Nash equilibria [8], and optimization problems [9–14], reduce to solving monotone inclusions. The simplest monotone inclusion is to find a zero point of a maximally monotone operator acting on a real Hilbert space. This problem

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can be solved efficiently by the proximal point algorithm when the resolvent of the operator is easy to implement numerically [15] (see [16–22] in the context of variable metric). The problem was then extended to the problem of finding a zero point of the sum of a maximally monotone operator and a cocoercive operator. In this case, we can use the forward-backward splitting algorithm [1,4,14,23] (see [24] in the context of variable metric). When the cocoercive operator is relaxed to be Lipschitzian and monotone, the problem can be solved by the forward-backward-forward splitting algorithm in [9,25] (see [26] for the variable metric extension of this algorithm). We also note that when the single-valued operator is replaced by any maximally monotone and multi-valued operator, we also have Douglas-Rachford splitting method; see [23] and the references therein.

The first composite monotone inclusion was studied in [9] for the sum of a composite operator and a maximally monotone operator. They have proposed a new primal-dual splitting algorithm to solve it as well as its dual problem. This framework was then extended to the inclusion with mixtures of composite, Lipschitzian, and parallel-sum type monotone operators [27]. In the special cases of [27], for instance, when the Lipschitzian monotone operator is either cocoercive or maximally monotone and multi-valued operator, alternative algorithms were also proposed [28], [29] (the variable metric versions of the algorithms in [27] and [28] were presented [26] and [24], respectively). Very recently, these frameworks have been unified into a system of monotone inclusions with mixtures of composite, Lipschitzian, and parallel-sum type monotone operators in [30].

Observe that when the problem has a structure, for examples, mixtures of composite, Lipschitzian or cocoercive, and parallel-sum type monotone operators as in [24,26–28], existing purely primal splitting methods do not offer satisfactory options to solve the problem due to the appearance of the composite components and hence alternative primal-dual strategies must be explored. In this paper, motivated by coupled monotone inclusions and systems of monotone inclusions, we address the numerical solutions of coupled system of primal-dual inclusions in real Hilbert spaces. We develop a flexible algorithm to solve this problem, analyze its asymptotic behavior, and show that it can solve several problems beyond the state of the art.

In the present paper, we state the coupled system of monotone inclusions under investigation and recall some notations and background on the monotone operator theory in Section 2. We propose a primal-dual splitting algorithm for solving this problem in Section 3. In Section 4, we provide application to coupled system of monotone inclusions in duality. Section 5 is devoted to applications to minimization problems. Section 6 is an application to multi-dictionary signal representation.

2 Problem Formulation and Technical Results

2.1 Problem Formulation

In this paper, we focus on solving the following coupled system of monotone inclusions.

Problem 2.1 Let m and s be strictly positive integers, let ν_0 and μ_0 be in $]0,+\infty[$. For every $i \in \{1,\ldots,m\}$, let $(\mathcal{H}_i, \langle\cdot \mid \cdot\rangle)$ be a real Hilbert space, let $z_i \in \mathcal{H}_i$, let $A_i\colon \mathcal{H}_i \rightrightarrows \mathcal{H}_i$ be maximally monotone, let $C_i\colon \mathcal{H}_1 \times \ldots \times \mathcal{H}_m \to \mathcal{H}_i$ be such that, for every $\boldsymbol{x} := (x_i)_{1\leq i\leq m}$ and $\boldsymbol{y} := (y_i)_{1\leq i\leq m}$ in $\mathcal{H}_1 \times \ldots \times \mathcal{H}_m$,

$$\sum_{i=1}^{m} \langle x_i - y_i \mid C_i\boldsymbol{x} - C_i\boldsymbol{y}\rangle \geq \nu_0 \sum_{i=1}^{m} \|C_i\boldsymbol{x} - C_i\boldsymbol{y}\|^2. \tag{1}$$

For every $k \in \{1,\ldots,s\}$, let $(\mathcal{G}_k, \langle\cdot \mid \cdot\rangle)$ be a real Hilbert space, let $r_k \in \mathcal{G}_k$, let $B_k\colon \mathcal{G}_k \rightrightarrows \mathcal{G}_k$ be maximally monotone, let $S_k\colon \mathcal{G}_1 \times \ldots \times \mathcal{G}_s \to \mathcal{G}_k$ be such that, for every $\boldsymbol{v} := (v_k)_{1\leq k\leq s}$ and $\boldsymbol{w} := (w_k)_{1\leq k\leq s}$ in $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$,

$$\sum_{k=1}^{s} \langle v_k - w_k \mid S_k\boldsymbol{v} - S_k\boldsymbol{w}\rangle \geq \mu_0 \sum_{k=1}^{s} \|S_k\boldsymbol{v} - S_k\boldsymbol{w}\|^2. \tag{2}$$

For every $i \in \{1,\ldots,m\}$ and for every $k \in \{1,\ldots,s\}$, let $L_{k,i}\colon \mathcal{H}_i \to \mathcal{G}_k$ be a bounded linear operator. The problem is to find $\overline{\boldsymbol{x}} := (\overline{x}_1,\ldots,\overline{x}_m)$ in $\mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ and $\overline{\boldsymbol{v}} := (\overline{v}_1,\ldots,\overline{v}_s)$ in $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$ such that

$$\begin{cases} z_1 - \displaystyle\sum_{k=1}^{s} L_{k,1}^*\overline{v}_k \in A_1\overline{x}_1 + C_1\overline{\boldsymbol{x}} \\ \vdots \\ z_m - \displaystyle\sum_{k=1}^{s} L_{k,m}^*\overline{v}_k \in A_m\overline{x}_m + C_m\overline{\boldsymbol{x}} \end{cases} \quad \text{and} \quad \begin{cases} \displaystyle\sum_{i=1}^{m} L_{1,i}\overline{x}_i - r_1 \in B_1\overline{v}_1 + S_1\overline{\boldsymbol{v}} \\ \vdots \\ \displaystyle\sum_{i=1}^{m} L_{s,i}\overline{x}_i - r_s \in B_s\overline{v}_s + S_s\overline{\boldsymbol{v}}. \end{cases} \tag{3}$$

We denote by Ω the set of solutions to (3).

In the case when $((L_{k,i})_{1\leq k\leq s})_{1\leq i\leq m}$ are zero, we can use the algorithm in [1] to solve the inclusions on the left hand side and the right hand side of (3) separately. Let us note that the non-linear coupling terms $(C_i)_{1\leq i\leq m}$ and $(S_k)_{1\leq k\leq s}$ were introduced in [1] and they are cocoercive operators which often play a central role; see for instance [1,3–7,14,23,31–33]. Let us add that the general algorithm in [30] can solve Problem 2.1 for the case when C_i and S_k are univariate, monotone and Lipschitzian. The algorithm proposed in [34] can solve Problem 2.1 for the case when $s = m$, and $(A_i)_{1\leq i\leq m}$ are zero, $(S_i)_{1\leq i\leq m}$ are univariate, and for each $(k,i) \in \{1,\ldots,m\}^2$ with $k \neq i$, $L_{k,i}$ is zero. Furthermore, the primal-dual algorithm in [24, Section 6] can solve Problem 2.1 for the case when $m = 1$ and each S_k is univariate, cocoercive.

To sum up, recent frameworks can solve special cases of Problem 2.1 and no existing algorithm can solve it in the general case.

Notation and background. Throughout, $\mathcal{H}$ and $\mathcal{G}$, and $(\mathcal{G}_i)_{1\leq i\leq m}$ are real Hilbert spaces. Their scalar products and associated norms are respectively denoted by $\langle\cdot\mid\cdot\rangle$ and $\|\cdot\|$. We denote by $\mathcal{B}\,(\mathcal{H},\mathcal{G})$ the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{G}$. The adjoint of $L\in\mathcal{B}\,(\mathcal{H},\mathcal{G})$ is denoted by L^*. We set $\mathcal{B}\,(\mathcal{H}):=\mathcal{B}\,(\mathcal{H},\mathcal{H})$. The symbols $\rightharpoonup$ and $\to$ denote respectively weak and strong convergence, and Id denotes the identity operator, we denote by $\ell^1_+(\mathbb{N})$ the set of summable sequences in $[0,+\infty[$ and by $\ell^2(\mathbb{K})$ $(\varnothing\neq\mathbb{K}\subset\mathbb{N})$ the set of square summable sequences, indexed by $\mathbb{K}$, in $\mathbb{R}$.

Let M_1 and M_2 be self-adjoint operators in $\mathcal{B}\,(\mathcal{H})$, we write

$$M_1\succcurlyeq M_2 \text{ iff } (\forall x\in\mathcal{H})\ \langle M_1x\mid x\rangle\geq\langle M_2x\mid x\rangle\,.$$

Let $\alpha\in\,]0,+\infty[$. We set

$$\mathcal{P}_\alpha(\mathcal{H}):=\big\{M\in\mathcal{B}\,(\mathcal{H}):M^*=M\quad\text{and}\quad M\succcurlyeq\alpha\,\mathrm{Id}\big\}.$$

The square root of M in $\mathcal{P}_\alpha(\mathcal{H})$ is denoted by $\sqrt{M}$. Moreover, for every M in $\mathcal{P}_\alpha(\mathcal{H})$, we define respectively a scalar product and a norm by

$$(\forall x\in\mathcal{H})(\forall y\in\mathcal{H})\quad\langle x\mid y\rangle_M:=\langle Mx\mid y\rangle\quad\text{and}\quad\|x\|_M:=\sqrt{\langle Mx\mid x\rangle}.$$

Let $A\colon\mathcal{H}\rightrightarrows\mathcal{H}$ be a set-valued operator. The domain and the graph of A are respectively defined by

$$\operatorname{dom}A:=\big\{x\in\mathcal{H}:Ax\neq\varnothing\big\}\text{ and }\operatorname{gra}A:=\big\{(x,u)\in\mathcal{H}\times\mathcal{H}:u\in Ax\big\}.$$

The set of zeros and the range of A are respectively defined by

$$\operatorname{zer}A:=\big\{x\in\mathcal{H}:0\in Ax\big\}\text{ and }\operatorname{ran}A:=\big\{u\in\mathcal{H}:(\exists\,x\in\mathcal{H})\;u\in Ax\big\}.$$

The inverse of A and the resolvent of A are respectively defined by

$$A^{-1}\colon\mathcal{H}\rightrightarrows\mathcal{H}\colon u\mapsto\big\{x\in\mathcal{H}:u\in Ax\big\}\text{ and }J_A:=(\mathrm{Id}+A)^{-1}.$$

Moreover, A is monotone iff

$$(\forall(x,y)\in\mathcal{H}\times\mathcal{H})(\forall(u,v)\in Ax\times Ay)\quad\langle x-y\mid u-v\rangle\geq0,$$

and maximally monotone iff it is monotone and there exists no monotone operator $B\colon\mathcal{H}\rightrightarrows\mathcal{H}$ such that $\operatorname{gra}A\subset\operatorname{gra}B$ and $A\neq B$. A single-valued operator $B\colon\mathcal{H}\to\mathcal{H}$ is β-cocoercive, for some $\beta\in\,]0,+\infty[$, iff

$$(\forall x\in\mathcal{H})(\forall y\in\mathcal{H})\quad\langle x-y\mid Bx-By\rangle\geq\beta\|Bx-By\|^2.$$

The parallel sum of $A\colon\mathcal{H}\rightrightarrows\mathcal{H}$ and $B\colon\mathcal{H}\rightrightarrows\mathcal{H}$ is

$$A\,\square\,B:=(A^{-1}+B^{-1})^{-1}.$$

Let $\Gamma_0(\mathcal{H})$ be the class of proper lower semicontinuous convex functions from $\mathcal{H}$ to $]-\infty,+\infty]$. For any $U \in \mathcal{P}_\alpha(\mathcal{H})$ and $f \in \Gamma_0(\mathcal{H})$, we define

$$J_{U^{-1}\partial f} := \mathrm{prox}_f^U : \mathcal{H} \to \mathcal{H} \colon x \mapsto \underset{y\in\mathcal{H}}{\mathrm{argmin}} \big(f(y) + \frac{1}{2}\|x-y\|_U^2\big), \tag{4}$$

and

$$J_{\partial f} := \mathrm{prox}_f \colon \mathcal{H} \to \mathcal{H} \colon x \mapsto \underset{y\in\mathcal{H}}{\mathrm{argmin}} \big(f(y) + \frac{1}{2}\|x-y\|^2\big),$$

and the conjugate function of f is

$$f^* \colon a \mapsto \sup_{x\in\mathcal{H}} \big(\langle a \mid x\rangle - f(x)\big).$$

Note that,

$$(\forall f \in \Gamma_0(\mathcal{H}))(x \in \mathcal{H})(y \in \mathcal{H}) \quad y \in \partial f(x) \Leftrightarrow x \in \partial f^*(y),$$

or equivalently,

$$(\forall f \in \Gamma_0(\mathcal{H})) \quad (\partial f)^{-1} = \partial f^*. \tag{5}$$

The infimal convolution of the two functions f and g from $\mathcal{H}$ to $]-\infty,+\infty]$ is

$$f \,\square\, g \colon x \mapsto \inf_{y\in\mathcal{H}} (f(y) + g(x-y)).$$

The indicator function of a nonempty, closed and convex set C is denoted by ι_C, its conjugate is the support function σ_C, the distance function of C is denoted by d_C. Finally, the strong relative interior of a subset C of $\mathcal{H}$ is the set of points $x \in C$ such that the cone generated by $-x+C$ is a closed vector subspace of $\mathcal{H}$.

2.2 Technical Results

We recall some results on monotone operators.

Definition 2.1 [1, Definition 2.3] An operator $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ is *demiregular* at $x \in \mathrm{dom}\, A$ iff, for every sequence $((x_n,u_n))_{n\in\mathbb{N}}$ in $\mathrm{gra}\, A$ and every $u \in Ax$ such that $x_n \rightharpoonup x$ and $u_n \to u$, we have $x_n \to x$.

Lemma 2.1 [1, Proposition 2.4] *Let $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be monotone and suppose that $x \in \mathrm{dom}\, A$. Then, A is demiregular at x in each of the following cases.*

(a) A is uniformly monotone at x, i.e., there exists an increasing function $\phi\colon [0,+\infty[\to [0,+\infty]$ that vanishes only at 0 such that

$$(\forall u \in Ax)(\forall (y,v) \in \mathrm{gra}\, A) \quad \langle x-y \mid u-v\rangle \geq \phi(\|x-y\|).$$

(b) A is strongly monotone, i.e., there exists $\alpha \in\,]0,+\infty[$ such that $A - \alpha\,\mathrm{Id}$ is monotone.

(c) *J_A is compact, i.e., for every bounded set $C \subset \mathcal{H}$, the closure of $J_A(C)$ is compact. In particular,* $\operatorname{dom} A$ *is boundedly relatively compact, i.e., the intersection of its closure with every closed ball is compact.*
(d) *$A\colon \mathcal{H} \to \mathcal{H}$ is single-valued with a single-valued continuous inverse.*
(e) *A is single-valued on* $\operatorname{dom} A$ *and* $\operatorname{Id} - A$ *is demicompact, i.e., for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in* $\operatorname{dom} A$ *such that $(Ax_n)_{n\in\mathbb{N}}$ converges strongly, $(x_n)_{n\in\mathbb{N}}$ admits a strong cluster point.*
(f) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ is uniformly convex at x, i.e., there exists an increasing function $\phi\colon [0,+\infty[\to [0,+\infty]$ that vanishes only at 0 such that, for every $\alpha \in]0,1[$ and every $y \in \operatorname{dom} f$,*

$$f\big(\alpha x + (1-\alpha)y\big) + \alpha(1-\alpha)\phi(\|x-y\|) \leq \alpha f(x) + (1-\alpha)f(y).$$

(g) *$A = \partial f$, where $f \in \Gamma_0(\mathcal{H})$ and, for every $\xi \in \mathbb{R}$, $\big\{x \in \mathcal{H} : f(x) \leq \xi\big\}$ is boundedly compact.*

Lemma 2.2 [24, Lemma 3.7] *Let $A\colon \mathcal{H} \rightrightarrows \mathcal{H}$ be maximally monotone, let $\alpha \in]0,+\infty[$, let $U \in \mathcal{P}_\alpha(\mathcal{H})$, and let $\mathcal{G}$ be the real Hilbert space obtained by endowing $\mathcal{H}$ with the scalar product $(x,y) \mapsto \langle x \mid y\rangle_{U^{-1}} := \big\langle x \mid U^{-1}y\big\rangle$. Then, the following hold.*

(a) *$UA\colon \mathcal{G} \rightrightarrows \mathcal{G}$ is maximally monotone.*
(b) *$J_{UA}\colon \mathcal{G} \to \mathcal{G}$ is 1-cocoercive, i.e., firmly nonexpansive, hence nonexpansive.*
(c) *$J_{UA} = (U^{-1} + A)^{-1} \circ U^{-1}$.*

Lemma 2.3 *Let α and β be strictly positive reals, let $B\colon \mathcal{H} \to \mathcal{H}$ be β-cocoercive, let $U \in \mathcal{P}_\alpha(\mathcal{H})$ be such that $\|U^{-1}\| < 2\beta$ and set $P := \operatorname{Id} - U^{-1}B$. Then,*

$$(\forall x \in \mathcal{H})(y \in \mathcal{H}) \quad \|Px - Py\|_U^2 \leq \|x-y\|_U^2 - (2\beta - \|U^{-1}\|)\|Bx - By\|^2. \tag{6}$$

Hence, P is nonexpansive with respect to the norm $\|\cdot\|_U$.

Proof. Let $x \in \mathcal{H}$ and $y \in \mathcal{H}$. Then, using the cocoercivity of B, we have

$$\begin{aligned}\|Px - Py\|_U^2 &= \|x-y\|_U^2 - 2\langle x - y \mid Bx - By\rangle + \|U^{-1}(Bx - By)\|_U^2 \\ &\leq \|x-y\|_U^2 - 2\beta\|Bx - By\|^2 + \big\langle Bx - By \mid U^{-1}(Bx - By)\big\rangle \\ &\leq \|x-y\|_U^2 - (2\beta - \|U^{-1}\|)\|Bx - By\|^2,\end{aligned}$$

which proves (6). ◻

Theorem 2.1 [24, Theorem 4.1] *Let $\boldsymbol{\mathcal{K}}$ be a real Hilbert space with scalar product $\langle\langle \cdot \mid \cdot \rangle\rangle$ and the associated norm $|||\cdot|||$. Let $A\colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}}$ be maximally monotone and $B\colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}}$ be β-cocoercive such that*

$$Z := \operatorname{zer}(A + B) \neq \varnothing. \tag{7}$$

Let $\alpha \in]0,+\infty[$, let $\beta \in]0,+\infty[$, let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence in $\ell_+^1(\mathbb{N})$, and let $(U_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\boldsymbol{\mathcal{K}})$ such that

$$\mu := \sup_{n\in\mathbb{N}} \|U_n\| < +\infty \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad (1+\eta_n)U_{n+1} \succcurlyeq U_n. \tag{8}$$

Let $\varepsilon \in \left]0, \min\{1, 2\beta/(\mu+1)\}\right]$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, (2\beta-\varepsilon)/\mu]$, let $x_0 \in \boldsymbol{\mathcal{K}}$, and let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be absolutely summable sequences in $\boldsymbol{\mathcal{K}}$. Then, iterate, for every $n \in \mathbb{N}$,

$$\left\lfloor \begin{array}{ll} 1. & y_n := x_n - \gamma_n U_n(Bx_n + b_n) \\ 2. & x_{n+1} := x_n + \lambda_n\big(J_{\gamma_n U_n A}(y_n) + a_n - x_n\big). \end{array} \right. \tag{9}$$

Then, the following hold for some $\overline{x} \in Z$.

(a) $x_n \rightharpoonup \overline{x}$.
(b) $\sum_{n\in\mathbb{N}} |||Bx_n - B\overline{x}|||^2 < +\infty$.
(c) Suppose that at every point in Z, A or B is demiregular, then $x_n \to \overline{x}$.

3 Algorithm and Convergence

We propose the following algorithm for solving Problem 2.1.

Algorithm 3.1 *Let $\alpha \in \left]0, +\infty\right[$ and, for every $i \in \{1, \dots, m\}$ and every $k \in \{1, \dots, s\}$, let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H}_i)$ and let $(V_{k,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_k)$. Set $\beta := \min\{\nu_0, \mu_0\}$, and let $\varepsilon \in \left]0, \min\{1, \beta\}\right[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$. Fix $(x_{i,0})_{1\le i\le m}$ in $\mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ and $(v_{k,0})_{1\le k\le s}$ in $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$. Then, iterate, for every $n \in \mathbb{N}$,*

$$\left\lfloor \begin{array}{l} (i)\ \text{For } i = 1, \dots, m \\ \left\lfloor \begin{array}{l} 1.\ t_{i,n} := \sum_{k=1}^{s} L_{k,i}^* v_{k,n} + C_{i,n}(x_{1,n}, \dots, x_{m,n}) + c_{i,n} \\ 2.\ p_{i,n} := J_{U_{i,n}A_i}\Big(x_{i,n} - U_{i,n}(t_{i,n} - z_i)\Big) + a_{i,n} \\ 3.\ y_{i,n} := 2p_{i,n} - x_{i,n} \\ 4.\ x_{i,n+1} := x_{i,n} + \lambda_n(p_{i,n} - x_{i,n}) \end{array} \right. \\ (ii)\ \text{For } k = 1, \dots, s \\ \left\lfloor \begin{array}{l} 1.\ w_{k,n} := \sum_{i=1}^{m} L_{k,i} y_{i,n} - S_{k,n}(v_{1,n}, \dots, v_{s,n}) - d_{k,n} \\ 2.\ q_{k,n} := J_{V_{k,n}B_k}\Big(v_{k,n} + V_{k,n}\big(w_{k,n} - r_k\big)\Big) + b_{k,n} \\ 3.\ v_{k,n+1} := v_{k,n} + \lambda_n(q_{k,n} - v_{k,n}), \end{array} \right. \end{array} \right. \tag{10}$$

where, for every $i \in \{1, \dots, m\}$ and every $k \in \{1, \dots, s\}$, the following conditions hold.

(a) $(\forall n \in \mathbb{N})$ $U_{i,n+1} \succeq U_{i,n}$ *and* $V_{k,n+1} \succeq V_{k,n}$, *and*

$$\mu := \sup_{n\in\mathbb{N}}\{\|U_{1,n}\|, \dots, \|U_{m,n}\|, \|V_{1,n}\|, \dots, \|V_{s,n}\|\} < +\infty.$$

(b) $(C_{i,n})_{n\in\mathbb{N}}$ *are operators from $\mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ to $\mathcal{H}_i$ such that*

(b1) $(C_{i,n} - C_i)_{n\in\mathbb{N}}$ *are Lipschitz continuous with respective constants* $(\kappa_{i,n})_{n\in\mathbb{N}} \in \left]0, +\infty\right[$ *satisfying*

$$\sum_{n\in\mathbb{N}} \kappa_{i,n} < +\infty.$$

(b2) There exists $\overline{\mathbf{s}} \in \mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ *not depending on* i *such that*

$$(\forall n \in \mathbb{N}) \quad C_{i,n}\overline{\mathbf{s}} = C_i\overline{\mathbf{s}}.$$

(c) $(S_{k,n})_{n\in\mathbb{N}}$ *are operators from* $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$ *to* $\mathcal{G}_k$ *such that*

(c1) $(S_{k,n}-S_k)_{n\in\mathbb{N}}$ *are Lipschitz continuous with respective constants* $(\eta_{k,n})_{n\in\mathbb{N}} \in \left]0,+\infty\right[$ *satisfying*

$$\sum_{n\in\mathbb{N}} \eta_{k,n} < +\infty.$$

(c2) There exists $\overline{\boldsymbol{w}} \in \mathcal{G}_1 \times \ldots \times \mathcal{G}_s$ *not depending on* k *such that*

$$(\forall n \in \mathbb{N}) \quad S_{k,n}\overline{\boldsymbol{w}} = S_k\overline{\boldsymbol{w}}.$$

(d) $(a_{i,n})_{n\in\mathbb{N}}$ *and* $(c_{i,n})_{n\in\mathbb{N}}$ *are absolutely summable sequences in* $\mathcal{H}_i$.

(e) $(b_{k,n})_{n\in\mathbb{N}}$ *and* $(d_{k,n})_{n\in\mathbb{N}}$ *are absolutely summable sequences in* $\mathcal{G}_k$.

Remark 3.1 Here are some remarks.

(a) Our algorithm has basically a structure of the variable metric forward-backward splitting since the multi-valued operators are used individually in the backward steps via their resolvents, the single-valued operators are used individually in the forward steps via their approximation values.

(b) The algorithm allows the metric to vary over the course of the iterations. Even when restricted to the constant metric case (which is the case where $(U_{i,n})_{1\le i\le m}$ and $(V_{k,n})_{1\le k\le s}$ are identity operators), the algorithm is new.

(c) Condition (a) was used in [24,26], while conditions (b) and (c) were used in [1], and conditions (d) and (e) which quantify the tolerance allowed in the inexact implementation of the resolvents and the approximations of single-valued operators are widely used in the literature.

(d) Algorithm 3.1 is an extension of [24, Corollary 6.2] where $m = 1$ and every $n \in \mathbb{N}$: $C_{1,n} = C$ and for every $k \in \{1,\ldots,s\}$, $S_{k,n} = D_k^{-1}$ is restricted to be univariate and cocoercive, and B_k is replaced by B_k^{-1}.

The main result of the paper can be now stated.

Theorem 3.1 *Suppose in Problem 2.1 that* $\Omega \neq \varnothing$ *and there exists* $L_{k_0,i_0} \neq 0$ *for some* $i_0 \in \{1,\ldots,m\}$ *and* $k_0 \in \{1,\ldots,s\}$. *For every* $n \in \mathbb{N}$, *set*

$$\delta_n := \left(\sqrt{\sum_{i=1}^{m}\sum_{k=1}^{s}\left\|\sqrt{V_{k,n}}L_{k,i}\sqrt{U_{i,n}}\right\|^2}\right)^{-1} - 1, \tag{11}$$

and suppose that

$$\zeta_n := \frac{\delta_n}{(1+\delta_n)\max_{1\le i\le m, 1\le k\le s}\{\|U_{i,n}\|, \|V_{k,n}\|\}} \geq \frac{1}{2\beta - \varepsilon}. \tag{12}$$

For every $i \in \{1,\ldots,m\}$ *and every* $k \in \{1,\ldots,s\}$, *let* $(x_{i,n})_{n\in\mathbb{N}}$ *and* $(v_{k,n})_{n\in\mathbb{N}}$ *be sequences generated by Algorithm 3.1. Then, the following hold for some* $(\overline{x}_1,\ldots,\overline{x}_m, \overline{v}_1,\ldots,\overline{v}_s) \in \Omega$.

(a) $(\forall i \in \{1,\ldots,m\})\ \ x_{i,n} \rightharpoonup \overline{x}_i$ *and* $(\forall k \in \{1,\ldots,s\})\ \ v_{k,n} \rightharpoonup \overline{v}_k$.

(b) *Suppose that the operator* $(x_i)_{1\leq i\leq m} \mapsto (C_j(x_i)_{1\leq i\leq m})_{1\leq j\leq m}$ *is demiregular (see Lemma 2.1 for special cases) at* $(\overline{x}_1,\ldots,\overline{x}_m)$*, then* $(\forall i \in \{1,\ldots,m\})$ $x_{i,n} \to \overline{x}_i$.

(c) *Suppose that the operator* $(v_k)_{1\leq k\leq s} \mapsto (S_j(v_k)_{1\leq k\leq s})_{1\leq j\leq s}$ *is demiregular (see Lemma 2.1 for special cases) at* $(\overline{v}_1,\ldots,\overline{v}_s)$*, then* $(\forall k \in \{1,\ldots,s\})$ $v_{k,n} \to \overline{v}_k$.

(d) *Suppose that there exists* $j \in \{1,\ldots,m\}$ *and an operator* $C\colon \mathcal{H}_j \to \mathcal{H}_j$ *such that* $(\forall (x_i)_{1\leq i\leq m} \in (\mathcal{H}_i)_{1\leq i\leq m})\ C_j(x_1,\ldots,x_m) = Cx_j$ *and* C *is demiregular (see Lemma 2.1 for special cases) at* $\overline{x}_j$*, then* $x_{j,n} \to \overline{x}_j$.

(e) *Suppose that there exists* $j \in \{1,\ldots,s\}$ *and an operator* $D\colon \mathcal{G}_j \to \mathcal{G}_j$ *such that* $(\forall (v_k)_{1\leq k\leq s} \in (\mathcal{G}_k)_{1\leq k\leq s})\ S_j(v_1,\ldots,v_s) = Dv_j$ *and* D *is demiregular (see Lemma 2.1 for special cases) at* $\overline{v}_j$*, then* $v_{j,n} \to \overline{v}_j$.

Proof. Let us introduce the Hilbert direct sums

$$\boldsymbol{\mathcal{H}} := \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_m, \quad \boldsymbol{\mathcal{G}} := \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_s, \quad \text{and} \quad \boldsymbol{\mathcal{K}} := \boldsymbol{\mathcal{H}} \oplus \boldsymbol{\mathcal{G}}.$$

We denote by $\boldsymbol{x} := (x_i)_{1\leq i\leq m}$, $\boldsymbol{y} := (y_i)_{1\leq i\leq m}$ the generic elements in $\boldsymbol{\mathcal{H}}$, and by $\boldsymbol{v} := (v_k)_{1\leq k\leq s}$, $\boldsymbol{w} := (w_k)_{1\leq k\leq s}$ the generic elements in $\boldsymbol{\mathcal{G}}$. The generic elements in $\boldsymbol{\mathcal{K}}$ will be denoted by $\boldsymbol{p} := (\boldsymbol{x},\boldsymbol{v})$. The scalar product and the norm of $\boldsymbol{\mathcal{H}}$ are respectively defined by

$$\langle \cdot \mid \cdot \rangle : (\boldsymbol{x},\boldsymbol{y}) \mapsto \sum_{i=1}^{m} \langle x_i \mid y_i \rangle \quad \text{and} \quad \| \cdot \| \colon \boldsymbol{x} \mapsto \sqrt{\langle \boldsymbol{x} \mid \boldsymbol{x} \rangle}.$$

The scalar product and the norm of $\boldsymbol{\mathcal{G}}$ are defined by the same fashion as those of $\boldsymbol{\mathcal{H}}$,

$$\langle \cdot \mid \cdot \rangle : (\boldsymbol{v},\boldsymbol{w}) \mapsto \sum_{k=1}^{s} \langle v_k \mid w_k \rangle \quad \text{and} \quad \| \cdot \| \colon \boldsymbol{v} \mapsto \sqrt{\langle \boldsymbol{v} \mid \boldsymbol{v} \rangle}.$$

We next define respectively the scalar product and the norm of $\boldsymbol{\mathcal{K}}$ by

$$\langle\langle \cdot \mid \cdot \rangle\rangle \colon \big((\boldsymbol{x},\boldsymbol{v}),(\boldsymbol{y},\boldsymbol{w})\big) \mapsto \sum_{i=1}^{m} \langle x_i \mid y_i \rangle + \sum_{k=1}^{s} \langle v_k \mid w_k \rangle$$

and

$$||| \cdot ||| \colon (\boldsymbol{x},\boldsymbol{v}) \mapsto \sqrt{\langle\langle (\boldsymbol{x},\boldsymbol{v}) \mid (\boldsymbol{x},\boldsymbol{v}) \rangle\rangle}. \tag{13}$$

Set

$$\begin{cases} \boldsymbol{A}\colon \boldsymbol{\mathcal{H}} \rightrightarrows \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto \bigtimes_{i=1}^{m} A_i x_i \\ \boldsymbol{B}\colon \boldsymbol{\mathcal{G}} \rightrightarrows \boldsymbol{\mathcal{G}}\colon \boldsymbol{v} \mapsto \bigtimes_{k=1}^{s} B_k v_k \\ \boldsymbol{C}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}}\colon \boldsymbol{x} \mapsto (C_i \boldsymbol{x})_{1\leq i\leq m} \\ \boldsymbol{D}\colon \boldsymbol{\mathcal{G}} \rightrightarrows \boldsymbol{\mathcal{G}}\colon \boldsymbol{v} \mapsto (S_k \boldsymbol{v})_{1\leq k\leq s} \\ \boldsymbol{L}\colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}}\colon \boldsymbol{x} \mapsto \big(\sum_{i=1}^{m} L_{k,i} x_i\big)_{1\leq k\leq s} \\ \boldsymbol{z} := (z_1,\ldots,z_m) \\ \boldsymbol{r} := (r_1,\ldots,r_s), \end{cases} \tag{14}$$

and for every $n \in \mathbb{N}$,

$$\boldsymbol{C}_n \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{x} \mapsto (C_{i,n}\boldsymbol{x})_{1\leq i\leq m} \quad \text{and} \quad \boldsymbol{D}_n \colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{G}} \colon \boldsymbol{v} \mapsto (S_{k,n}\boldsymbol{v})_{1\leq k\leq s}.$$

Then, it follows from (1) that

$$(\forall \boldsymbol{x} \in \boldsymbol{\mathcal{H}})(\boldsymbol{y} \in \boldsymbol{\mathcal{H}}) \quad \langle \boldsymbol{x} - \boldsymbol{y} \mid \boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\boldsymbol{y}\rangle \geq \nu_0\|\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\boldsymbol{y}\|^2, \tag{15}$$

from (2) that

$$(\forall \boldsymbol{v} \in \boldsymbol{\mathcal{G}})(\boldsymbol{w} \in \boldsymbol{\mathcal{G}}) \quad \langle \boldsymbol{v} - \boldsymbol{w} \mid \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{w}\rangle \geq \mu_0\|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{w}\|^2, \tag{16}$$

which shows that $\boldsymbol{C}$ and $\boldsymbol{D}$ are respectively ν_0-cocoercive and μ_0-cocoercive and hence they are maximally monotone [6, Example 20.28]. Moreover, it follows from [6, Proposition 20.23] that $\boldsymbol{A}$ and $\boldsymbol{B}$ are maximally monotone. Furthermore,

$$\boldsymbol{L}^* \colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{H}} \colon \boldsymbol{v} \mapsto \left(\sum_{k=1}^{s} L_{k,i}^* v_k\right)_{1\leq i\leq m}.$$

Then, using (14), we can rewrite the system of monotone inclusions (3) as a monotone inclusion in $\boldsymbol{\mathcal{K}}$,

$$\text{find } (\overline{\boldsymbol{x}}, \overline{\boldsymbol{v}}) \in \boldsymbol{\mathcal{K}} \text{ such that } \big(\boldsymbol{z} - \boldsymbol{L}^*\overline{\boldsymbol{v}}, \boldsymbol{L}\overline{\boldsymbol{x}} - \boldsymbol{r}\big) \in \Big((\boldsymbol{A}+\boldsymbol{C})\overline{\boldsymbol{x}}, (\boldsymbol{B}+\boldsymbol{D})\overline{\boldsymbol{v}}\Big). \tag{17}$$

Set

$$\begin{cases} \boldsymbol{M} \colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}} \colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (-\boldsymbol{z} + \boldsymbol{A}\boldsymbol{x}, \boldsymbol{r} + \boldsymbol{B}\boldsymbol{v}) \\ \boldsymbol{S} \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{L}^*\boldsymbol{v}, -\boldsymbol{L}\boldsymbol{x}) \\ \boldsymbol{Q} \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{C}\boldsymbol{x}, \boldsymbol{D}\boldsymbol{v}), \end{cases} \tag{18}$$

and for every $n \in \mathbb{N}$,

$$\begin{cases} \boldsymbol{Q}_n \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\boldsymbol{C}_n\boldsymbol{x}, \boldsymbol{D}_n\boldsymbol{v}) \\ \boldsymbol{U}_n \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto \big((U_{i,n}x_i)_{1\leq i\leq m}, (V_{k,n}v_k)_{1\leq k\leq s}\big) \\ \boldsymbol{V}_n \colon \boldsymbol{\mathcal{K}} \to \boldsymbol{\mathcal{K}} \colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto \boldsymbol{U}_n^{-1}(\boldsymbol{x}, \boldsymbol{v}) - (\boldsymbol{L}^*\boldsymbol{v}, \boldsymbol{L}\boldsymbol{x}). \end{cases} \tag{19}$$

Then, $\boldsymbol{M}$ and $\boldsymbol{S}$ are maximally monotone, and (15), (16) imply that $\boldsymbol{Q}$ is β-cocoercive and hence it is maximally monotone [6, Example 20.28]. Therefore, $\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q}$ is maximally monotone [6, Corollary 24.4]. Furthermore, the problem (17) reduces to find a zero point of $\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q}$. Note that $\Omega \neq \varnothing$ implies that

$$\operatorname{zer}(\boldsymbol{M} + \boldsymbol{S} + \boldsymbol{Q}) \neq \varnothing. \tag{20}$$

We next derive from the condition (a) in Algorithm 3.1 that

$$\mu := \sup_{n\in\mathbb{N}} \|\boldsymbol{U}_n\| < +\infty, \quad \text{and} \quad (\forall n \in \mathbb{N}) \quad \boldsymbol{U}_{n+1} \succeq \boldsymbol{U}_n \in \mathcal{P}_\alpha(\boldsymbol{\mathcal{K}}), \tag{21}$$

and it follows from (13) and [19, Lemma 2.1(ii)] that, for every $n \in \mathbb{N}$ and every $\boldsymbol{p} := (\boldsymbol{x}, \boldsymbol{v}) \in \boldsymbol{\mathcal{K}}$,

$$
\begin{aligned}
|||\boldsymbol{p}|||^2_{\boldsymbol{U}_n^{-1}} &= \sum_{i=1}^m \|x_i\|^2_{U_{i,n}^{-1}} + \sum_{k=1}^s \|v_k\|^2_{V_{k,n}^{-1}} \\
&\geq \sum_{i=1}^m \|x_i\|^2 \|U_{i,n}^{-1}\| + \sum_{k=1}^s \|v_k\|^2 \|V_{k,n}^{-1}\| \\
&\geq \left(\sum_{i=1}^m \|x_i\|^2 + \sum_{k=1}^s \|v_k\|^2 \right) \min_{1 \leq i \leq m, 1 \leq k \leq s} \{\|U_{i,n}\|^{-1}, \|V_{k,n}\|^{-1}\} \\
&= |||\boldsymbol{p}|||^2 \min_{1 \leq i \leq m, 1 \leq k \leq s} \{\|U_{i,n}\|^{-1}, \|V_{k,n}\|^{-1}\}.
\end{aligned} \tag{22}
$$

Note that $(\boldsymbol{V}_n)_{n\in\mathbb{N}}$ are self-adjoint, let us check that $(\boldsymbol{V}_n)_{n\in\mathbb{N}}$ are strongly monotone. To this end, let us introduce

$$
(\forall n \in \mathbb{N}) \begin{cases} \boldsymbol{T}_n \colon \boldsymbol{\mathcal{H}} \to \boldsymbol{\mathcal{G}} \colon \boldsymbol{x} \mapsto \left(\sum_{i=1}^m \sqrt{V_{k,n}} L_{k,i} x_i \right)_{1 \leq k \leq s} \\ \\ \boldsymbol{R}_n \colon \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{G}} \colon \boldsymbol{v} \mapsto \left(\sqrt{V_{1,n}}^{\,-1} v_1, \ldots, \sqrt{V_{s,n}}^{\,-1} v_s \right). \end{cases} \tag{23}
$$

We note that, for every $i \in \{1, \ldots, m\}$ and every $n \in \mathbb{N}$,

$$
(\forall x \in \mathcal{H}_i) \quad \left\langle x \mid U_{i,n}^{-1}\, x \right\rangle = \left\langle \sqrt{U_{i,n}}^{\,-1} x \mid \sqrt{U_{i,n}}^{\,-1} x \right\rangle = \|\sqrt{U_{i,n}}^{\,-1} x\|^2.
$$

Then, by using Cauchy-Schwarz's inequality, for every $n \in \mathbb{N}$ and every $\boldsymbol{x} \in \boldsymbol{\mathcal{H}}$, we have

$$
\begin{aligned}
\|\boldsymbol{T}_n \boldsymbol{x}\|^2 &= \sum_{k=1}^s \left\| \sum_{i=1}^m \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \sqrt{U_{i,n}}^{\,-1} x_i \right\|^2 \\
&\leq \sum_{k=1}^s \left(\sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\| \left\| \sqrt{U_{i,n}}^{\,-1} x_i \right\| \right)^2 \\
&\leq \sum_{k=1}^s \left(\sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\|^2 \right) \left(\sum_{i=1}^m \left\| \sqrt{U_{i,n}}^{\,-1} x_i \right\|^2 \right) \\
&= \left(\sum_{i=1}^m \|x_i\|^2_{U_{i,n}^{-1}} \right) \sum_{k=1}^s \sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\|^2 \\
&= \beta_n \sum_{i=1}^m \|x_i\|^2_{U_{i,n}^{-1}},
\end{aligned} \tag{24}
$$

where we set

$$
(\forall n \in \mathbb{N}) \quad \beta_n := \sum_{k=1}^s \sum_{i=1}^m \left\| \sqrt{V_{k,n}} L_{k,i} \sqrt{U_{i,n}} \right\|^2,
$$

which together with (11) imply that

$$(\forall n\in\mathbb{N})\quad (1+\delta_n)\beta_n=\frac{1}{1+\delta_n}. \tag{25}$$

Moreover,

$$\begin{aligned}(\forall n\in\mathbb{N})(\boldsymbol{v}\in\boldsymbol{\mathcal{G}})\quad \|\boldsymbol{R}_n\boldsymbol{v}\|^2&=\sum_{k=1}^{s}\big\|\sqrt{V_{k,n}}^{\,-1}v_k\big\|^2\\&=\sum_{k=1}^{s}\|v_k\|^2_{V_{k,n}^{-1}}.\end{aligned}\tag{26}$$

Therefore, for every $\boldsymbol{p}:=(\boldsymbol{x},\boldsymbol{v})\in\boldsymbol{\mathcal{K}}$ and every $n\in\mathbb{N}$, it follows from (19), (23), (25), (26) and (22), (12) that

$$\begin{aligned}\langle\langle\boldsymbol{p}\mid\boldsymbol{V}_n\boldsymbol{p}\rangle\rangle&=|||\boldsymbol{p}|||^2_{\boldsymbol{U}_n^{-1}}-2\left\langle\boldsymbol{L}\boldsymbol{x}\mid\boldsymbol{v}\right\rangle\\&=|||\boldsymbol{p}|||^2_{\boldsymbol{U}_n^{-1}}-2\sum_{k=1}^{s}\left\langle\sum_{i=1}^{m}\sqrt{V_{k,n}}L_{k,i}x_i\mid\sqrt{V_{k,n}}^{-1}v_k\right\rangle\\&=|||\boldsymbol{p}|||^2_{\boldsymbol{U}_n^{-1}}-2\left\langle\sqrt{(1+\delta_n)\beta_n}^{\,-1}\boldsymbol{T}_n\boldsymbol{x}\mid\sqrt{(1+\delta_n)\beta_n}\boldsymbol{R}_n\boldsymbol{v}\right\rangle\\&\geq|||\boldsymbol{p}|||^2_{\boldsymbol{U}_n^{-1}}-\left(\frac{|||\boldsymbol{T}_n\boldsymbol{x}|||^2}{(1+\delta_n)\beta_n}+(1+\delta_n)\beta_n\|\boldsymbol{R}_n\boldsymbol{v}\|^2\right)\\&\geq|||\boldsymbol{p}|||^2_{\boldsymbol{U}_n^{-1}}-\left(\frac{\sum_{i=1}^m\|x_i\|^2_{U_{i,n}^{-1}}}{(1+\delta_n)}+(1+\delta_n)\beta_n\sum_{k=1}^{s}\|v_k\|^2_{V_{k,n}^{-1}}\right)\\&=\frac{\delta_n}{1+\delta_n}\left(\sum_{i=1}^{m}\|x_i\|^2_{U_{i,n}^{-1}}+\sum_{k=1}^{s}\|v_k\|^2_{V_{k,n}^{-1}}\right)\\&\geq\zeta_n|||\boldsymbol{p}|||^2.\end{aligned}\tag{27}$$

In turn, $(\boldsymbol{V}_n)_{n\in\mathbb{N}}$ are invertible, by [19, Lemma 2.1(iii)] and (12),

$$(\forall n\in\mathbb{N})\quad \|\boldsymbol{V}_n^{-1}\|\leq\frac{1}{\zeta_n}\leq 2\beta-\varepsilon,$$

and by [19, Lemma 2.1(i)], (21), for every $n\in\mathbb{N}$,

$$\boldsymbol{U}_{n+1}\succcurlyeq\boldsymbol{U}_n\Rightarrow\boldsymbol{U}_n^{-1}\succcurlyeq\boldsymbol{U}_{n+1}^{-1}\Rightarrow\boldsymbol{V}_n\succcurlyeq\boldsymbol{V}_{n+1}\Rightarrow\boldsymbol{V}_{n+1}^{-1}\succcurlyeq\boldsymbol{V}_n^{-1}.$$

Furthermore, for every $\boldsymbol{p}\in\boldsymbol{\mathcal{K}}$, we derive from [19, Lemma 2.1(ii)] that

$$\langle\langle\boldsymbol{V}_n^{-1}\boldsymbol{p}\mid\boldsymbol{p}\rangle\rangle\geq\|\boldsymbol{V}_n\|^{-1}|||\boldsymbol{p}|||^2\geq\frac{1}{\rho}|||\boldsymbol{p}|||^2,\ \text{ where }\rho:=\alpha^{-1}+\|\boldsymbol{S}\|. \tag{28}$$

Altogether,

$$\sup_{n\in\mathbb{N}}\|\boldsymbol{V}_n^{-1}\|\leq 2\beta-\varepsilon\quad\text{and}\quad(\forall n\in\mathbb{N})\quad\boldsymbol{V}_{n+1}^{-1}\succcurlyeq\boldsymbol{V}_n^{-1}\in\mathcal{P}_{1/\rho}(\boldsymbol{\mathcal{K}}). \tag{29}$$

Moreover, using [19, Lemma 2.1(i)(ii)] and (29), we obtain

$$\big(\forall(\boldsymbol{z}_n)_{n\in\mathbb{N}}\in\boldsymbol{\mathcal{K}}^{\mathbb{N}}\big)\quad \sum_{n\in\mathbb{N}}|||\boldsymbol{z}_n|||<+\infty\quad\Leftrightarrow\quad\sum_{n\in\mathbb{N}}|||\boldsymbol{z}_n|||_{\boldsymbol{V}_n^{-1}}<+\infty \tag{30}$$

and

$$\big(\forall(\boldsymbol{z}_n)_{n\in\mathbb{N}}\in\boldsymbol{\mathcal{K}}^{\mathbb{N}}\big)\quad \sum_{n\in\mathbb{N}}|||\boldsymbol{z}_n|||<+\infty\quad\Leftrightarrow\quad\sum_{n\in\mathbb{N}}|||\boldsymbol{z}_n|||_{\boldsymbol{V}_n}<+\infty, \tag{31}$$

and

$$(\forall\boldsymbol{p}\in\boldsymbol{\mathcal{K}})\quad\sup_{n\in\mathbb{N}}|||\boldsymbol{p}|||_{\boldsymbol{V}_n}<+\infty. \tag{32}$$

Now we can reformulate the algorithm (10) as iterations in the space $\boldsymbol{\mathcal{K}}$. We first observe that (10) is equivalent to

$$\left\lfloor\begin{array}{l}(i)\ \text{For}\ i=1,\ldots,m\\ \quad\left\lfloor\begin{array}{l}1.\ U_{i,n}^{-1}(x_{i,n}-p_{i,n})-\sum_{k=1}^{s}L_{k,i}^{*}v_{k,n}-C_{i,n}(x_{1,n},\ldots,x_{m,n})\in\\ \qquad\qquad -z_i+A_i(p_{i,n}-a_{i,n})+c_{i,n}-U_{i,n}^{-1}a_{i,n}\\ 2.\ x_{i,n+1}=x_{i,n}+\lambda_n(p_{i,n}-x_{i,n})\end{array}\right.\\ (ii)\ \text{For}\ k=1,\ldots,s\\ \quad\left\lfloor\begin{array}{l}1.\ V_{k,n}^{-1}(v_{k,n}-q_{k,n})-\sum_{i=1}^{m}L_{k,i}(x_{i,n}-p_{i,n})-S_{k,n}(v_{1,n},\ldots,v_{s,n})\in\\ \qquad r_k+B_k(q_{k,n}-b_{k,n})-\sum_{i=1}^{m}L_{k,i}p_{i,n}+d_{k,n}-V_{k,n}^{-1}b_{k,n}\\ 2.\ v_{k,n+1}=v_{k,n}+\lambda_n(q_{k,n}-v_{k,n}).\end{array}\right.\end{array}\right. \tag{33}$$

Set

$$\begin{cases}\boldsymbol{p}_n:=(x_{1,n},\ldots,x_{m,n},v_{1,n},\ldots,v_{s,n})\\ \boldsymbol{y}_n:=(p_{1,n},\ldots,p_{m,n},q_{1,n},\ldots,q_{s,n})\\ \boldsymbol{a}_n:=(a_{1,n},\ldots,a_{m,n},b_{1,n},\ldots,b_{s,n})\\ \boldsymbol{c}_n:=(c_{1,n},\ldots,c_{m,n},d_{1,n},\ldots,d_{s,n})\\ \boldsymbol{d}_n:=\big((U_{i,n}^{-1}a_{i,n})_{1\le i\le m},(V_{k,n}^{-1}b_{k,n})_{1\le k\le m}\big)\\ \boldsymbol{b}_n:=(\boldsymbol{S}+\boldsymbol{V}_n)\boldsymbol{a}_n+\boldsymbol{c}_n-\boldsymbol{d}_n.\end{cases}$$

Then, using the same argument as in [28, Eqs. (3.22)–(3.35)], using (18), (19), (33) yields

$$(\forall n\in\mathbb{N})\quad\left\lfloor\begin{array}{l}\boldsymbol{V}_n(\boldsymbol{p}_n-\boldsymbol{y}_n)-\boldsymbol{Q}_n\boldsymbol{p}_n\in(\boldsymbol{M}+\boldsymbol{S})(\boldsymbol{y}_n-\boldsymbol{a}_n)+\boldsymbol{S}\boldsymbol{a}_n+\boldsymbol{c}_n-\boldsymbol{d}_n\\ \boldsymbol{p}_{n+1}=\boldsymbol{p}_n+\lambda_n(\boldsymbol{y}_n-\boldsymbol{p}_n).\end{array}\right. \tag{34}$$

For every $n\in\mathbb{N}$, we have

$$\begin{aligned}&\boldsymbol{V}_n(\boldsymbol{p}_n-\boldsymbol{y}_n)-\boldsymbol{Q}_n\boldsymbol{p}_n\in(\boldsymbol{M}+\boldsymbol{S})(\boldsymbol{y}_n-\boldsymbol{a}_n)+\boldsymbol{S}\boldsymbol{a}_n+\boldsymbol{c}_n-\boldsymbol{d}_n\\ \Leftrightarrow\ &(\boldsymbol{V}_n-\boldsymbol{Q}_n)\boldsymbol{p}_n\in(\boldsymbol{M}+\boldsymbol{S}+\boldsymbol{V}_n)(\boldsymbol{y}_n-\boldsymbol{a}_n)+(\boldsymbol{S}+\boldsymbol{V}_n)\boldsymbol{a}_n+\boldsymbol{c}_n-\boldsymbol{d}_n\\ \Leftrightarrow\ &\boldsymbol{y}_n=\big(\boldsymbol{M}+\boldsymbol{S}+\boldsymbol{V}_n\big)^{-1}\Big((\boldsymbol{V}_n-\boldsymbol{Q}_n)\boldsymbol{p}_n-(\boldsymbol{S}+\boldsymbol{V}_n)\boldsymbol{a}_n-\boldsymbol{c}_n+\boldsymbol{d}_n\Big)+\boldsymbol{a}_n\\ \Leftrightarrow\ &\boldsymbol{y}_n=\Big(\mathbf{Id}+\boldsymbol{V}_n^{-1}(\boldsymbol{M}+\boldsymbol{S})\Big)^{-1}\Big(\big(\mathbf{Id}-\boldsymbol{V}_n^{-1}\boldsymbol{Q}_n\big)\boldsymbol{p}_n-\boldsymbol{V}_n^{-1}\boldsymbol{b}_n\Big)+\boldsymbol{a}_n\\ \Leftrightarrow\ &\boldsymbol{y}_n=J_{\boldsymbol{V}_n^{-1}(\boldsymbol{M}+\boldsymbol{S})}\Big(\big(\mathbf{Id}-\boldsymbol{V}_n^{-1}\boldsymbol{Q}_n\big)\boldsymbol{p}_n-\boldsymbol{V}_n^{-1}\boldsymbol{b}_n\Big)+\boldsymbol{a}_n.\end{aligned} \tag{35}$$

Therefore, for every $n \in \mathbb{N}$, (34) becomes

$$\boldsymbol{p}_{n+1} = \boldsymbol{p}_n + \lambda_n \Big(J_{\boldsymbol{V}_n^{-1}(\boldsymbol{M}+\boldsymbol{S})} \Big(\boldsymbol{p}_n - \boldsymbol{V}_n^{-1}(\boldsymbol{Q}_n \boldsymbol{p}_n + \boldsymbol{b}_n) \Big) + \boldsymbol{a}_n - \boldsymbol{p}_n \Big). \tag{36}$$

By setting, for every $n \in \mathbb{N}$,

$$\begin{cases} \widetilde{\boldsymbol{M}} \colon \boldsymbol{\mathcal{K}} \rightrightarrows \boldsymbol{\mathcal{K}} \colon (\boldsymbol{x}, \boldsymbol{v}) \mapsto \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{v}) + \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{v}), \\ \widetilde{\boldsymbol{P}}_n := \mathrm{Id} - \boldsymbol{V}_n^{-1} \boldsymbol{Q}_n \quad \text{and} \quad \boldsymbol{P}_n := \mathrm{Id} - \boldsymbol{V}_n^{-1} \boldsymbol{Q}, \\ \boldsymbol{E}_n := \boldsymbol{Q}_n - \boldsymbol{Q} \quad \text{and} \quad \widetilde{\boldsymbol{Q}}_n := \boldsymbol{V}_n^{-1} \boldsymbol{E}_n, \\ \boldsymbol{e}_{1,n} := \widetilde{\boldsymbol{Q}}_n \boldsymbol{p}_n + \boldsymbol{V}_n^{-1} \boldsymbol{b}_n, \end{cases} \tag{37}$$

we have

$$\begin{aligned} (36) \Leftrightarrow \boldsymbol{p}_{n+1} &= \boldsymbol{p}_n + \lambda_n \Big(J_{\boldsymbol{V}_n^{-1} \widetilde{\boldsymbol{M}}} \big(\widetilde{\boldsymbol{P}}_n \boldsymbol{p}_n - \boldsymbol{V}_n^{-1} \boldsymbol{b}_n \big) + \boldsymbol{a}_n - \boldsymbol{p}_n \Big) \\ &= (1-\lambda_n) \boldsymbol{p}_n + \lambda_n J_{\boldsymbol{V}_n^{-1} \widetilde{\boldsymbol{M}}} \big(\widetilde{\boldsymbol{P}}_n \boldsymbol{p}_n - \boldsymbol{V}_n^{-1} \boldsymbol{b}_n \big) + \lambda_n \boldsymbol{a}_n \\ &= (1-\lambda_n) \boldsymbol{p}_n + \lambda_n J_{\boldsymbol{V}_n^{-1} \widetilde{\boldsymbol{M}}} \big(\boldsymbol{P}_n \boldsymbol{p}_n - \boldsymbol{e}_{1,n} \big) + \lambda_n \boldsymbol{a}_n \qquad (38) \\ &= (1-\lambda_n) \boldsymbol{p}_n + \lambda_n \Big(J_{\boldsymbol{V}_n^{-1} \widetilde{\boldsymbol{M}}} \big(\boldsymbol{P}_n \boldsymbol{p}_n - \boldsymbol{e}_{1,n} \big) + \boldsymbol{a}_n \Big). \qquad (39) \end{aligned}$$

Algorithm (39) is a special instance of the variable metric forward-backward splitting (9) with

$$(\forall n \in \mathbb{N}) \ \gamma_n = 1 \in \Big[\varepsilon, (2\beta - \varepsilon)/(\sup_{n \in \mathbb{N}} \|\boldsymbol{V}_n^{-1}\|) \Big] \ \text{(see (29))}.$$

Note that $\widetilde{\boldsymbol{M}}$ is maximally monotone, $\boldsymbol{Q}$ is β-cocoercive, and $(\lambda_n)_{n\in\mathbb{N}}$ in $[\varepsilon, 1]$. Moreover, (29) and (20) show respectively that (8) and (7) are satisfied. Therefore, in view of Theorem 2.1, it is sufficient to prove that $(\boldsymbol{e}_{1,n})_{n\in\mathbb{N}}$ and $(\boldsymbol{e}_n)_{n\in\mathbb{N}}$ are absolutely summable in $\boldsymbol{\mathcal{K}}$, i.e, we prove that

$$\sum_{n\in\mathbb{N}} |||\boldsymbol{e}_{1,n}||| < +\infty, \tag{40}$$

and

$$\sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| < +\infty. \tag{41}$$

For every $i \in \{1, \ldots, m\}$ and every $k \in \{1, \ldots, s\}$, since $(a_{i,n})_{n\in\mathbb{N}}$, $(c_{i,n})_{n\in\mathbb{N}}$ and $(b_{k,n})_{n\in\mathbb{N}}$ and $(d_{k,n})_{n\in\mathbb{N}}$ are absolutely summable, we have

$$\begin{cases} \displaystyle\sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| \leq \sum_{n\in\mathbb{N}} \left(\sum_{i=1}^m \|a_{i,n}\| + \sum_{k=1}^s \|b_{k,n}\| \right) < +\infty, \\ \displaystyle\sum_{n\in\mathbb{N}} |||\boldsymbol{c}_n||| \leq \sum_{n\in\mathbb{N}} \left(\sum_{i=1}^m \|c_{i,n}\| + \sum_{k=1}^s \|d_{k,n}\| \right) < +\infty. \end{cases} \tag{42}$$

Moreover, for every $n \in \mathbb{N}$, $\boldsymbol{U}_n \in \mathcal{P}_\alpha(\boldsymbol{\mathcal{K}})$, it follows from [19, Lemma 2.1(iii)] that $\|\boldsymbol{U}_n^{-1}\| \leq \alpha^{-1}$. Hence,

$$\begin{cases} \sum_{n\in\mathbb{N}} |||\boldsymbol{d}_n||| \leq \alpha^{-1} \sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| < +\infty, \\ \sum_{n\in\mathbb{N}} |||\boldsymbol{b}_n||| \leq (\rho + \|\boldsymbol{S}\|) \sum_{n\in\mathbb{N}} |||\boldsymbol{a}_n||| + \sum_{n\in\mathbb{N}} \big(|||\boldsymbol{c}_n||| + |||\boldsymbol{d}_n|||\big) < +\infty. \end{cases}$$

Therefore, $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$, $(\boldsymbol{c}_n)_{n\in\mathbb{N}}$ and $(\boldsymbol{d}_n)_{n\in\mathbb{N}}$ are absolutely summable in $\boldsymbol{\mathcal{K}}$. Next it follows from the conditions (b), (c) in Algorithm 3.1 and (29), (27), (12) that, for every $\boldsymbol{p} := (\boldsymbol{x}, \boldsymbol{v}) \in \boldsymbol{\mathcal{K}}$ and every $\boldsymbol{q} := (\boldsymbol{y}, \boldsymbol{w}) \in \boldsymbol{\mathcal{K}}$, and every $n \in \mathbb{N}$,

$$\begin{aligned} |||\widetilde{\boldsymbol{Q}}_n\boldsymbol{p} - \widetilde{\boldsymbol{Q}}_n\boldsymbol{q}|||^2_{\boldsymbol{V}_n} &= \langle\langle \widetilde{\boldsymbol{Q}}_n\boldsymbol{p} - \widetilde{\boldsymbol{Q}}_n\boldsymbol{q} \mid \boldsymbol{V}_n\Big(\widetilde{\boldsymbol{Q}}_n\boldsymbol{p} - \widetilde{\boldsymbol{Q}}_n\boldsymbol{q}\Big)\rangle\rangle \\ &= \langle\langle \boldsymbol{E}_n\boldsymbol{p} - \boldsymbol{E}_n\boldsymbol{q} \mid \boldsymbol{V}_n^{-1}\boldsymbol{E}_n\boldsymbol{p} - \boldsymbol{V}_n^{-1}\boldsymbol{E}_n\boldsymbol{q}\rangle\rangle \\ &\leq \|\boldsymbol{V}_n^{-1}\| \; |||\boldsymbol{E}_n\boldsymbol{p} - \boldsymbol{E}_n\boldsymbol{q}|||^2 \\ &\leq (2\beta - \varepsilon)\Bigg(\|(\boldsymbol{C}_n - \boldsymbol{C})\boldsymbol{x} - (\boldsymbol{C}_n - \boldsymbol{C})\boldsymbol{y}\|^2 \\ &\qquad\qquad + \|(\boldsymbol{D}_n - \boldsymbol{D})\boldsymbol{v} - (\boldsymbol{D}_n - \boldsymbol{D})\boldsymbol{w}\|^2\Bigg) \\ &= (2\beta - \varepsilon)\Bigg(\sum_{i=1}^m \|(C_{i,n} - C_i)\boldsymbol{x} - (C_{i,n} - C_i)\boldsymbol{y}\|^2 \\ &\qquad\qquad + \sum_{k=1}^s \|(S_{k,n} - S_k)\boldsymbol{v} - (S_{k,n} - S_k)\boldsymbol{w}\|^2\Bigg) \\ &\leq (2\beta - \varepsilon)\Bigg(\sum_{i=1}^m \kappa_{i,n}^2 \|\boldsymbol{x} - \boldsymbol{y}\|^2 + \sum_{k=1}^s \eta_{k,n}^2 \|\boldsymbol{v} - \boldsymbol{w}\|^2\Bigg) \\ &\leq (2\beta - \varepsilon)\Bigg(\sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2\Bigg) |||\boldsymbol{p} - \boldsymbol{q}|||^2 \\ &\leq (2\beta - \varepsilon)\zeta_n^{-1}\Bigg(\sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2\Bigg) |||\boldsymbol{p} - \boldsymbol{q}|||^2_{\boldsymbol{V}_n} \\ &\leq (2\beta - \varepsilon)^2\Bigg(\sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2\Bigg) |||\boldsymbol{p} - \boldsymbol{q}|||^2_{\boldsymbol{V}_n}, \end{aligned} \tag{43}$$

which implies that $\widetilde{\boldsymbol{Q}}_n$ is Lipschitz continuous (in the norm $|||\cdot|||_{\boldsymbol{V}_n}$) with respectively constant

$$\kappa_n := (2\beta - \varepsilon)\sqrt{\sum_{i=1}^m \kappa_{i,n}^2 + \sum_{k=1}^s \eta_{k,n}^2},$$

that satisfies

$$\sum_{n\in\mathbb{N}} \kappa_n < +\infty. \tag{44}$$

Let $\boldsymbol{p} := (\boldsymbol{x}, \boldsymbol{v}) \in \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S}+\boldsymbol{Q})$ and noting that $(\forall n \in \mathbb{N})\ \widetilde{\boldsymbol{Q}}_n(\overline{\boldsymbol{s}}, \overline{\boldsymbol{w}}) = 0$,

$$\begin{aligned}
|||\boldsymbol{e}_{1,n}|||_{\boldsymbol{V}_n} &\le |||\widetilde{\boldsymbol{Q}}_n\boldsymbol{p}_n|||_{\boldsymbol{V}_n} + |||\boldsymbol{V}_n^{-1}\boldsymbol{b}_n|||_{\boldsymbol{V}_n}\\
&\le |||\widetilde{\boldsymbol{Q}}_n\boldsymbol{p}_n - \widetilde{\boldsymbol{Q}}_n\boldsymbol{p}|||_{\boldsymbol{V}_n} + |||\widetilde{\boldsymbol{Q}}_n\boldsymbol{p} - \widetilde{\boldsymbol{Q}}_n(\overline{\boldsymbol{s}}, \overline{\boldsymbol{w}})|||_{\boldsymbol{V}_n} + |||\boldsymbol{V}_n^{-1}\boldsymbol{b}_n|||_{\boldsymbol{V}_n}\\
&\le \kappa_n|||\boldsymbol{p}_n - \boldsymbol{p}|||_{\boldsymbol{V}_n} + \kappa_n|||\boldsymbol{p} - (\overline{\boldsymbol{s}}, \overline{\boldsymbol{w}})|||_{\boldsymbol{V}_n} + |||\boldsymbol{V}_n^{-1}\boldsymbol{b}_n|||_{\boldsymbol{V}_n}\\
&= \kappa_n|||\boldsymbol{p}_n - \boldsymbol{p}|||_{\boldsymbol{V}_n} + \kappa_n|||\boldsymbol{p} - (\overline{\boldsymbol{s}}, \overline{\boldsymbol{w}})|||_{\boldsymbol{V}_n} + |||\boldsymbol{b}_n|||_{\boldsymbol{V}_n^{-1}}.
\end{aligned} \tag{45}$$

Since $\boldsymbol{p} \in \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S}+\boldsymbol{Q})$, we have

$$(\forall n \in \mathbb{N}) \quad \boldsymbol{p} = J_{\boldsymbol{V}_n^{-1}\widetilde{\boldsymbol{M}}}(\boldsymbol{P}_n\boldsymbol{p}).$$

Hence, for each $n \in \mathbb{N}$, since $J_{\boldsymbol{V}_n^{-1}\widetilde{\boldsymbol{M}}}$ and $\boldsymbol{P}_n$ are nonexpansive with respect to the norm $|||\cdot|||_{\boldsymbol{V}_n}$ by Lemma 2.2(b) and Lemma 2.3, we have

$$\begin{aligned}
|||J_{\boldsymbol{V}_n^{-1}\widetilde{\boldsymbol{M}}}\big(\boldsymbol{P}_n\boldsymbol{p}_n - \boldsymbol{e}_{1,n}\big) - \boldsymbol{p}|||_{\boldsymbol{V}_n} &= |||J_{\boldsymbol{V}_n^{-1}\widetilde{\boldsymbol{M}}}\big(\boldsymbol{P}_n\boldsymbol{p}_n - \boldsymbol{e}_{1,n}\big)\\
&\qquad - J_{\boldsymbol{V}_n^{-1}\widetilde{\boldsymbol{M}}}(\boldsymbol{P}_n\boldsymbol{p})|||_{\boldsymbol{V}_n}\\
&\le |||\boldsymbol{p}_n - \boldsymbol{p}|||_{\boldsymbol{V}_n} + |||\boldsymbol{e}_{1,n}|||_{\boldsymbol{V}_n},
\end{aligned}$$

which and (38) imply that

$$\begin{aligned}
|||\boldsymbol{p}_{n+1} - \boldsymbol{p}|||_{\boldsymbol{V}_n} &\le \big|\big|\big|(1-\lambda_n)\big(\boldsymbol{p}_n - \boldsymbol{p}\big)\big|\big|\big|_{\boldsymbol{V}_n}\\
&\quad + \big|\big|\big|\lambda_n\big(J_{\boldsymbol{V}_n^{-1}\widetilde{\boldsymbol{M}}}\big(\boldsymbol{P}_n\boldsymbol{p}_n - \boldsymbol{e}_{1,n}\big) - \boldsymbol{p}\big)\big|\big|\big|_{\boldsymbol{V}_n} + \big|\big|\big|\lambda_n\boldsymbol{a}_n\big|\big|\big|_{\boldsymbol{V}_n}\\
&\le \Big(1 - \lambda_n + \lambda_n\Big)\big|\big|\boldsymbol{p}_n - \boldsymbol{p}\big|\big|_{\boldsymbol{V}_n} + |||\boldsymbol{e}_{1,n}|||_{\boldsymbol{V}_n}\\
&\qquad + \big|\big|\big|\lambda_n\boldsymbol{a}_n\big|\big|\big|_{\boldsymbol{V}_n}\\
&\le (1+\kappa_n)|||\boldsymbol{p}_n - \boldsymbol{p}|||_{\boldsymbol{V}_n} + \alpha_n,
\end{aligned} \tag{46}$$

where

$$(\forall n \in \mathbb{N}) \quad \alpha_n := \kappa_n|||\boldsymbol{p} - (\overline{\boldsymbol{s}}, \overline{\boldsymbol{w}})|||_{\boldsymbol{V}_n} + \big|\big|\big|\boldsymbol{b}_n\big|\big|\big|_{\boldsymbol{V}_n^{-1}} + \big|\big|\big|\boldsymbol{a}_n\big|\big|\big|_{\boldsymbol{V}_n}. \tag{47}$$

Noting that (32), (31), (30) and (43), (42), we have

$$\sum_{n\in\mathbb{N}} \alpha_n < +\infty. \tag{48}$$

Therefore, we derive from (46) and $(\forall n \in \mathbb{N})\ \boldsymbol{V}_n \succeq \boldsymbol{V}_{n+1}$ that

$$(\forall n \in \mathbb{N}) \quad \big|\big|\big|\boldsymbol{p}_{n+1} - \boldsymbol{p}\big|\big|\big|_{\boldsymbol{V}_{n+1}} \le (1+\kappa_n)|||\boldsymbol{p}_n - \boldsymbol{p}|||_{\boldsymbol{V}_n} + \alpha_n, \tag{49}$$

and hence, by [35, Lemma 2.2.2],

$$\sup_{n\in\mathbb{N}} |||\boldsymbol{p}_n - \boldsymbol{p}|||_{\boldsymbol{V}_n} < +\infty, \tag{50}$$

which and (45),(44),(48), (43), (30), (31), (32) imply that

$$\sum_{n\in\mathbb{N}} |||\boldsymbol{e}_{1,n}|||_{\boldsymbol{V}_n} < +\infty. \tag{51}$$

Therefore, (40) and (41) are proved.

(a): By Theorem 2.1(a), $\boldsymbol{p}_n \rightharpoonup \overline{\boldsymbol{p}} \in \operatorname{zer}(\boldsymbol{M}+\boldsymbol{S}+\boldsymbol{Q})$.

(b)(c): By Theorem 2.1(b) and (c),

$$|||\boldsymbol{Q}\boldsymbol{p}_n - \boldsymbol{Q}\overline{\boldsymbol{p}}||| \to 0,$$

which implies that, for every $i \in \{1,\ldots,m\}$ and every $k \in \{1,\ldots,s\}$,

$$\begin{cases} C_i(x_{1,n},\ldots,x_{m,n}) - C_i(\overline{x}_1,\ldots,\overline{x}_m) \to 0,\\ S_k(v_{1,n},\ldots,v_{s,n}) - S_k(\overline{v}_1,\ldots,\overline{v}_s) \to 0.\end{cases}$$

Moreover, by (a), we obtain, for every $i \in \{1,\ldots,m\}$, $x_{i,n} \rightharpoonup \overline{x}_i$ and for every $k \in \{1,\ldots,s\}$, $v_{k,n} \rightharpoonup \overline{v}_k$. Therefore, the conclusions follow from the definition of the demiregular operators.

(d)(e): The conclusions follow from our assumptions and the definition of the demiregular operators. □

4 Application to Coupled System of Monotone Inclusions in Duality

We provide an application to coupled system of monotone inclusions. Our problem formulation covers not only a wide class of monotone inclusions and duality frameworks in the literature (see [9,13,14,23,24,28,29,31,32,36–40] and the references therein), and coupled system of monotone inclusions unified in [1], but also a wide class of minimization formulations, in particular, in the multi-component signal decomposition and recovery; see [1,41,42] and the references therein.

Problem 4.1 Let m and s be strictly positive integers, and let ν_0 be in $]0,+\infty[$. For every $i \in \{1,\ldots,m\}$, let $(\mathcal{H}_i, \langle\cdot\mid\cdot\rangle)$ be a real Hilbert space, let $z_i \in \mathcal{H}_i$, let $A_i\colon \mathcal{H}_i \rightrightarrows \mathcal{H}_i$ be maximally monotone, let $C_i\colon \mathcal{H}_1\times\ldots\times\mathcal{H}_m \to \mathcal{H}_i$ be such that for every $\boldsymbol{x} := (x_i)_{1\le i\le m}$ and $\boldsymbol{y} := (y_i)_{1\le i\le m}$ in $\mathcal{H}_1\times\ldots\times\mathcal{H}_m$,

$$\sum_{i=1}^m \langle x_i - y_i \mid C_i\boldsymbol{x} - C_i\boldsymbol{y}\rangle \ge \nu_0 \sum_{i=1}^m \|C_i\boldsymbol{x} - C_i\boldsymbol{y}\|^2. \tag{52}$$

For every $k \in \{1,\ldots,s\}$, let $(\mathcal{G}_k, \langle\cdot\mid\cdot\rangle)$ be a real Hilbert space, let $r_k \in \mathcal{G}_k$, let $D_k\colon \mathcal{G}_k \rightrightarrows \mathcal{G}_k$ be maximally monotone and ν_k-strongly monotone for some ν_k in $]0,+\infty[$, let $B_k\colon \mathcal{G}_k \rightrightarrows \mathcal{G}_k$ be maximally monotone. For every $i \in \{1,\ldots,m\}$

and every $k \in \{1, \ldots, s\}$, let $L_{k,i} \colon \mathcal{H}_i \to \mathcal{G}_k$ be a bounded linear operator. The primal inclusion is to find $\overline{\boldsymbol{x}} := (\overline{x}_1, \ldots, \overline{x}_m)$ in $\mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ such that

$$
\begin{cases}
z_1 \in A_1\overline{x}_1 + \displaystyle\sum_{k=1}^{s} L_{k,1}^*\Big((D_k \,\square\, B_k)\Big(\sum_{i=1}^{m} L_{k,i}\overline{x}_i - r_k\Big)\Big) + C_1\overline{\boldsymbol{x}}\\
\vdots\\
z_m \in A_m\overline{x}_m + \displaystyle\sum_{k=1}^{s} L_{k,m}^*\Big((D_k \,\square\, B_k)\Big(\sum_{i=1}^{m} L_{k,i}\overline{x}_i - r_k\Big)\Big) + C_m\overline{\boldsymbol{x}}.
\end{cases} \tag{53}
$$

We denote by $\mathcal{P}$ the set of solutions to (53). The dual inclusion is to find $\overline{v}_1 \in \mathcal{G}_1, \ldots, \overline{v}_s \in \mathcal{G}_s$ such that

$$
\big(\exists \boldsymbol{x} := (x_i)_{1\le i\le m} \in (\mathcal{H}_i)_{1\le i\le m}\big)
\begin{cases}
z_1 - \displaystyle\sum_{k=1}^{s} L_{k,1}^*\overline{v}_k \in A_1x_1 + C_1\boldsymbol{x}\\
\vdots\\
z_m - \displaystyle\sum_{k=1}^{s} L_{k,m}^*\overline{v}_k \in A_mx_m + C_m\boldsymbol{x},
\end{cases}
\text{and}
\begin{cases}
\displaystyle\sum_{i=1}^{m} L_{1,i}x_i - r_1 \in B_1^{-1}\overline{v}_1 + D_1^{-1}\overline{v}_1\\
\vdots\\
\displaystyle\sum_{i=1}^{m} L_{s,i}x_i - r_s \in B_s^{-1}\overline{v}_s + D_s^{-1}\overline{v}_s.
\end{cases} \tag{54}
$$

The set of solutions to (54) is denoted by $\mathcal{D}$.

Remark 4.1 In Problem 4.1, there are two types of coupling. The first one is the smooth coupling modeled by $(C_i)_{1\le i\le m}$. The second one is the non-smooth coupling involving the parallel sums modeled by the second terms in (53). The frameworks in [30] and [34] consider respectively non-smooth coupling and smooth coupling only. The condition on $(C_i)_{1\le i\le m}$ is relaxed to be monotone and Lipschitzian in [34] but the algorithm proposed has an additional forward step even when $(C_i)_{1\le i\le m}$ are restricted to be cocoercive. This double coupling model can be easily solved by using our duality framework.

Algorithm 4.1 *Let $\alpha \in \left]0, +\infty\right[$ and, for every $i \in \{1, \ldots, m\}$ and every $k \in \{1, \ldots, s\}$, let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H}_i)$ and let $(V_{k,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_k)$. Set $\beta := \min\{\nu_0, \nu_1, \ldots, \nu_s\}$, and let $\varepsilon \in \left]0, \min\{1, \beta\}\right[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$. Let $(x_{i,0})_{1\le i\le m} \in \mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ and $(v_{k,0})_{1\le k\le s} \in \mathcal{G}_1 \times \ldots \times \mathcal{G}_s$. Then, iterate, for every $n \in \mathbb{N}$,*

$$
\left|
\begin{array}{l}
(i)\ \text{For } i = 1, \ldots, m\\
\left|
\begin{array}{l}
1.\ t_{i,n} := \sum_{k=1}^{s} L_{k,i}^* v_{k,n} + C_{i,n}(x_{1,n}, \ldots, x_{m,n}) + c_{i,n}\\
2.\ p_{i,n} := J_{U_{i,n}A_i}\big(x_{i,n} - U_{i,n}(t_{i,n} - z_i)\big) + a_{i,n}\\
3.\ y_{i,n} := 2p_{i,n} - x_{i,n}\\
4.\ x_{i,n+1} := x_{i,n} + \lambda_n(p_{i,n} - x_{i,n})
\end{array}
\right.\\
(ii)\ \text{For } k = 1, \ldots, s\\
\left|
\begin{array}{l}
1.\ w_{k,n} := \sum_{i=1}^{m} L_{k,i}y_{i,n} - \widetilde{S}_{k,n}v_{k,n} - d_{k,n}\\
2.\ q_{k,n} := J_{V_{k,n}B_k^{-1}}\big(v_{k,n} + V_{k,n}(w_{k,n} - r_k)\big) + b_{k,n}\\
3.\ v_{k,n+1} := v_{k,n} + \lambda_n(q_{k,n} - v_{k,n}),
\end{array}
\right.
\end{array}
\right. \tag{55}
$$

where, for every $i \in \{1,\dots,m\}$ and every $k \in \{1,\dots,s\}$, the following conditions hold.

(a) $(\forall n \in \mathbb{N})\ U_{i,n+1} \succeq U_{i,n}$ *and* $V_{k,n+1} \succeq V_{k,n}$, *and*

$$\mu := \sup_{n\in\mathbb{N}}\{\|U_{1,n}\|,\dots,\|U_{m,n}\|,\|V_{1,n}\|,\dots,\|V_{s,n}\|\} < +\infty.$$

(b) $(C_{i,n})_{n\in\mathbb{N}}$ *are operators from* $\mathcal{H}_1 \times \dots \times \mathcal{H}_m$ *to* $\mathcal{H}_i$ *such that*

(b1) $(C_{i,n}-C_i)_{n\in\mathbb{N}}$ *are Lipschitz continuous with respective constants* $(\kappa_{i,n})_{n\in\mathbb{N}} \in \left]0,+\infty\right[$ *satisfying*

$$\sum_{n\in\mathbb{N}} \kappa_{i,n} < +\infty.$$

(b2) There exists $\overline{\boldsymbol{s}} \in \mathcal{H}_1 \times \dots \times \mathcal{H}_m$ *not depending on* i *such that*

$$(\forall n \in \mathbb{N}) \quad C_{i,n}\overline{\boldsymbol{s}} = C_i\overline{\boldsymbol{s}}.$$

(c) $(\widetilde{S}_{k,n})_{n\in\mathbb{N}}$ *are operators from* $\mathcal{G}_k$ *to* $\mathcal{G}_k$ *such that*

(c1) $(\widetilde{S}_{k,n} - D_k^{-1})_{n\in\mathbb{N}}$ *are Lipschitz continuous with respective constants* $(\eta_{k,n})_{n\in\mathbb{N}} \in \left]0,+\infty\right[$ *satisfying*

$$\sum_{n\in\mathbb{N}} \eta_{k,n} < +\infty.$$

(c2) There exists $\overline{\boldsymbol{w}} := (\overline{w}_j)_{1\le j\le s} \in \mathcal{G}_1 \times \dots \times \mathcal{G}_s$ *not depending on* k *such that*

$$(\forall n \in \mathbb{N}) \quad \widetilde{S}_{k,n}\overline{w}_k = D_k^{-1}\overline{w}_k.$$

(d) $(a_{i,n})_{n\in\mathbb{N}}$ *and* $(c_{i,n})_{n\in\mathbb{N}}$ *are absolutely summable sequences in* $\mathcal{H}_i$.

(e) $(b_{k,n})_{n\in\mathbb{N}}$ *and* $(d_{k,n})_{n\in\mathbb{N}}$ *are absolutely summable sequences in* $\mathcal{G}_k$.

Corollary 4.1 *Suppose that* $\mathcal{P} \neq \varnothing$ *and there exists* $L_{k_0,i_0} \neq 0$, *for some* i_0 *in* $\{1,\dots,m\}$ *and* k_0 *in* $\{1,\dots,s\}$, *and* (12) *is satisfied. For every* $i \in \{1,\dots,m\}$ *and every* $k \in \{1,\dots,s\}$, *let* $(x_{i,n})_{n\in\mathbb{N}}$ *and* $(v_{k,n})_{n\in\mathbb{N}}$ *be sequences generated by Algorithm 4.1. Then, the following hold for some* $(\overline{x}_1,\dots,\overline{x}_m) \in \mathcal{P}$ *and* $(\overline{v}_1,\dots,\overline{v}_s) \in \mathcal{D}$.

(a) $(\forall i \in \{1,\dots,m\})\ \ x_{i,n} \rightharpoonup \overline{x}_i$ *and* $(\forall k \in \{1,\dots,s\})\ \ v_{k,n} \rightharpoonup \overline{v}_k$.

(b) Suppose that the operator $(x_i)_{1\le i\le m} \mapsto (C_j(x_i)_{1\le i\le m})_{1\le j\le m}$ *is demiregular (see Lemma 2.1 for special cases) at* $(\overline{x}_1,\dots,\overline{x}_m)$, *then* $(\forall i \in \{1,\dots,m\})$ $x_{i,n} \to \overline{x}_i$.

(c) Suppose that D_j^{-1} *is demiregular (see Lemma 2.1 for special cases) at* $\overline{v}_j$, *for some* $j \in \{1,\dots,s\}$, *then* $v_{j,n} \to \overline{v}_j$.

(d) Suppose that there exists $j \in \{1,\dots,m\}$ *and operator* $C\colon \mathcal{H}_j \to \mathcal{H}_j$ *such that* $(\forall (x_i)_{1\le i\le m} \in (\mathcal{H}_i)_{1\le i\le m})\ C_j(x_1,\dots,x_m) = Cx_j$ *and* C *is demiregular (see Lemma 2.1 for special cases) at* $\overline{x}_j$, *then* $x_{j,n} \to \overline{x}_j$.

Proof. Set $\mu_0 := \min\{\nu_1, \ldots, \nu_s\}$ and define

$$(\forall k \in \{1, \ldots, s\}) \quad S_k \colon \mathcal{G}_1 \times \ldots \times \mathcal{G}_s \to \mathcal{G}_k \colon (v_1, \ldots, v_s) \mapsto D_k^{-1} v_k. \tag{56}$$

Then, for every $\boldsymbol{v} := (v_k)_{1 \leq k \leq s}$ in $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$ and every $\boldsymbol{w} := (w_k)_{1 \leq k \leq s}$ in $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$, we obtain

$$\begin{aligned}
\sum_{k=1}^{s} \langle v_k - w_k \mid S_k \boldsymbol{v} - S_k \boldsymbol{w} \rangle &= \sum_{k=1}^{s} \left\langle v_k - w_k \mid D_k^{-1} v_k - D_k^{-1} w_k \right\rangle \\
&\geq \sum_{k=1}^{s} \nu_k \| D_k^{-1} v_k - D_k^{-1} w_k \|^2 \\
&\geq \mu_0 \sum_{k=1}^{s} \| D_k^{-1} v_k - D_k^{-1} w_k \|^2 \\
&= \mu_0 \sum_{k=1}^{s} \| S_k \boldsymbol{v} - S_k \boldsymbol{w} \|^2,
\end{aligned} \tag{57}$$

which shows that (2) is satisfied. Note that the conditions (a), (b), (d), and (e) in Algorithm 4.1 are the same as in Algorithm 3.1. Let us check the condition (c) in Algorithm 3.1. For every $n \in \mathbb{N}$, define

$$(\forall k \in \{1, \ldots, s\}) \quad S_{k,n} \colon \mathcal{G}_1 \times \ldots \times \mathcal{G}_s \to \mathcal{G}_k \colon (v_1, \ldots, v_s) \mapsto \widetilde{S}_{k,n} v_k. \tag{58}$$

Since $(\widetilde{S}_{k,n} - D_k^{-1})_{n \in \mathbb{N}}$ are Lipschitz continuous with respective constants $(\eta_{k,n})_{n \in \mathbb{N}}$, for every $\boldsymbol{v} := (v_k)_{1 \leq k \leq s}$ in $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$, $\boldsymbol{w} := (w_k)_{1 \leq k \leq s}$ in $\mathcal{G}_1 \times \ldots \times \mathcal{G}_s$, and every $n \in \mathbb{N}$, $k \in \{1, \ldots, s\}$, we have

$$\begin{aligned}
\| (S_{k,n} - S_k) \boldsymbol{v} - (S_{k,n} - S_k) \boldsymbol{w} \|^2 &= \| (\widetilde{S}_{k,n} - D_k^{-1}) v_k - (\widetilde{S}_{k,n} - D_k^{-1}) w_k \|^2 \\
&\leq \eta_{k,n}^2 \| v_k - w_k \|^2 \\
&\leq \eta_{k,n}^2 \sum_{j=1}^{s} \| v_j - w_j \|^2 \\
&= \eta_{k,n}^2 \| \boldsymbol{v} - \boldsymbol{w} \|^2.
\end{aligned}$$

Moreover, it follows from (56), (58) and the condition (c2) in Algorithm 4.1 that

$$(\forall n \in \mathbb{N})(k \in \{1, \ldots, s\}) \quad S_{k,n} \overline{\boldsymbol{w}} = \widetilde{S}_{k,n} \overline{w}_k = D_k^{-1} \overline{w}_k = S_k \overline{\boldsymbol{w}}.$$

Hence, the condition (c) in Algorithm 3.1 is also satisfied. Furthermore, the algorithm (10) reduces to (55) where B_k is replaced by B_k^{-1}. Next, since $\mathcal{P} \neq \varnothing$, we derive from (53) that, for every $k \in \{1, \ldots, s\}$, there exists $\overline{v}_k \in \mathcal{G}_k$ such that

$$\overline{v}_k \in (D_k \,\square\, B_k) \left(\sum_{i=1}^{m} L_{k,i} \overline{x}_i - r_k \right) \Leftrightarrow \sum_{i=1}^{m} L_{k,i} \overline{x}_i - r_k \in B_k^{-1} \overline{v}_k + D_k^{-1} \overline{v}_k, \tag{59}$$

and

$$(\forall i \in \{1,\dots,m\}) \quad z_i - \sum_{k=1}^{s} L_{k,i}^* \overline{v}_k \in A_i \overline{x}_i + C_i(\overline{x}_1,\dots,\overline{x}_m), \tag{60}$$

which show that $\Omega \neq \varnothing$ and $\mathcal{D} \neq \varnothing$. Inversely, if $(\overline{x}_1,\dots,\overline{x}_m,\overline{v}_1,\dots,\overline{v}_s) \in \Omega$, then the inclusions (59) and (60) are satisfied. Hence $(\overline{v}_1,\dots,\overline{v}_s) \in \mathcal{D}$ and $(\overline{x}_1,\dots,\overline{x}_m) \in \mathcal{P}$. Therefore, the conclusions follow from Theorem 3.1. ☐

5 Application to Minimization Problems

We provide applications to minimization problems involving infimal convolutions, composite functions and couplings. In the classic setting, the coupling is often modeled by the constraints of the form

$$(\forall k \in \{1,\dots,s\}) \ \sum_{i=1}^{m} L_{k,i} x_i \in D_k,$$

where $(D_k)_{1\leq k\leq s}$ are nonempty, closed and convex subsets of $\mathcal{G}_k$. The degree of violation of each hard constraint is measured by ι_{D_k}. However, due to the imprecise prior information or unmodeled dynamics in the data formation process, there are inaccuracies in the definition of the several sets in the family $(D_k)_{1\leq k\leq s}$. Therefore, instead of coupling by hard constraints, some authors use soft constraints by forcing $\sum_{i=1}^m L_{k,i}x_i$ to be close to D_k. This forcing is often measured by distance functions. This strategy was used in [41]. Here we follow the approach of [30], we use the parallel sums to model the coupling but we separate the smooth components and they are modeled by a differentiable function $\varphi \in \Gamma_0(\mathcal{H}_1 \times \ldots \times \mathcal{H}_m)$.

Problem 5.1 Let m and s be strictly positive integers. For every $i \in \{1,\dots,m\}$, let $\mathcal{H}_i$ be a real Hilbert space, let $z_i \in \mathcal{H}_i$, let $f_i \in \Gamma_0(\mathcal{H}_i)$. For every $k \in \{1,\dots,s\}$, let $\mathcal{G}_k$ be a real Hilbert space, let $r_k \in \mathcal{G}_k$, let $\ell_k \in \Gamma_0(\mathcal{G}_k)$ be ν_k-strongly convex function, for some $\nu_k \in \,]0,+\infty[$, let $g_k \in \Gamma_0(\mathcal{G}_k)$. For every $i \in \{1,\dots,m\}$ and every $k \in \{1,\dots,s\}$, let $L_{k,i}\colon \mathcal{H}_i \to \mathcal{G}_k$ be a bounded linear operator. Let $\varphi\colon \mathcal{H}_1 \times \ldots \times \mathcal{H}_m \to \mathbb{R}$ be convex differentiable function with ν_0^{-1}-Lipschitz continuous gradient. The primal problem is to

$$\underset{x_1\in\mathcal{H}_1,\dots,x_m\in\mathcal{H}_m}{\text{minimize}} \sum_{i=1}^{m} \big(f_i(x_i) - \langle x_i \mid z_i\rangle\big) + \sum_{k=1}^{s} \big(\ell_k \,\square\, g_k\big)\Big(\sum_{i=1}^{m} L_{k,i}x_i - r_k\Big) + \varphi(x_1,\dots,x_m), \tag{61}$$

under the assumption that there exists $\overline{\boldsymbol{x}} := (\overline{x}_1,\dots,\overline{x}_m)$ such that, for every $i \in \{1,\dots,m\}$,

$$z_i \in \partial f_i(\overline{x}_i) + \sum_{k=1}^{s} L_{k,i}^* \circ \Big(\partial\ell_k \,\square\, \partial g_k\Big) \circ \Big(\sum_{j=1}^{m} L_{k,j}\overline{x}_j - r_k\Big) + \nabla_i \varphi(\overline{\boldsymbol{x}}), \tag{62}$$

where $\nabla_i\varphi$ is the ith component of the gradient $\nabla\varphi$, and the dual problem is to

$$\underset{v_1\in\mathcal{G}_1,\ldots,v_s\in\mathcal{G}_s}{\text{minimize}} \left(\varphi^* \,\square\, \left(\sum_{i=1}^m f_i^*\right)\right)\left(\left(z_i - \sum_{k=1}^s L_{k,i}^* v_k\right)_{1\le i\le m}\right) + \sum_{k=1}^s \left(\ell_k^*(v_k) + g_k^*(v_k) + \langle v_k \mid r_k\rangle\right). \tag{63}$$

In the case when the infimal convolutions are absent, Problem 5.1 often appears in the multi-components signal decomposition and recovery problems; see [1, 8, 41] and the references therein.

Example 5.1 Some special cases of this problem are listed in the following.

(a) In the case when $\varphi\colon (x_1,\ldots,x_m)\mapsto \sum_{i=1}^m h_i(x_i)$, where, for every i in $\{1,\ldots,m\}$, $h_i\colon \mathcal{H}_i\to\mathbb{R}$ is a convex differentiable function with τ_i^{-1}-Lipschitz continuous gradient, for some $\tau_i\in\,]0,+\infty[$, Problem 5.1 reduces to the general minimization problem [30, Problem 5.1] which covers a wide class of the convex minimization problems in the literature.
(b) In the case when $\varphi\colon (x_1,\ldots,x_m)\mapsto 0$ and, for every $k\in\{1,\ldots,s\}$, $\ell_k := \iota_{\{0\}}$ and g_k is a convex differentiable function with τ_k^{-1}-Lipschitz continuous gradient, for some $\tau_k\in\,]0,+\infty[$, Problem 5.1 reduces to [41, Problem 1.1].
(c) In the case when $m=1$, Problem 5.1 reduces to [27, Problem 4.1] which was also studied in [24, 28].

Algorithm 5.1 *Let $\alpha\in\,]0,+\infty[$ and, for every $i\in\{1,\ldots,m\}$ and every $k\in\{1,\ldots,s\}$, let $(U_{i,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{H}_i)$ and let $(V_{k,n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{P}_\alpha(\mathcal{G}_k)$. Set $\beta := \min\{\nu_0,\nu_1,\ldots,\nu_s\}$, and let $\varepsilon\in\,]0,\min\{1,\beta\}[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon,1]$. Let $(x_{i,0})_{1\le i\le m}\in\mathcal{H}_1\times\ldots\times\mathcal{H}_m$ and $(v_{k,0})_{1\le k\le s}\in\mathcal{G}_1\times\ldots\times\mathcal{G}_s$. Then, iterate, for every $n\in\mathbb{N}$,*

$$\left\lfloor\begin{array}{l}
(i)\ \text{For } i=1,\ldots,m\\
\left\lfloor\begin{array}{l}
1.\ t_{i,n} := \sum_{k=1}^s L_{k,i}^* v_{k,n} + \nabla_i\varphi_n(x_{1,n},\ldots,x_{m,n}) + c_{i,n}\\
2.\ p_{i,n} := \operatorname{prox}_{f_i}^{U_{i,n}^{-1}}\big(x_{i,n} - U_{i,n}(t_{i,n}-z_i)\big) + a_{i,n}\\
3.\ y_{i,n} := 2p_{i,n} - x_{i,n}\\
4.\ x_{i,n+1} := x_{i,n} + \lambda_n(p_{i,n} - x_{i,n})
\end{array}\right.\\
(ii)\ \text{For } k=1,\ldots,s\\
\left\lfloor\begin{array}{l}
1.\ w_{k,n} := \sum_{i=1}^m L_{k,i} y_{i,n} - \nabla\widetilde{\ell}_{k,n}(v_{k,n}) - d_{k,n}\\
2.\ q_{k,n} := \operatorname{prox}_{g_k^*}^{V_{k,n}^{-1}}\big(v_{k,n} + V_{k,n}(w_{k,n} - r_k)\big) + b_{k,n}\\
3.\ v_{k,n+1} := v_{k,n} + \lambda_n(q_{k,n} - v_{k,n}),
\end{array}\right.
\end{array}\right. \tag{64}$$

where, for every $i\in\{1,\ldots,m\}$ and every $k\in\{1,\ldots,s\}$, the following conditions hold.

(a) $(\forall n \in \mathbb{N})$ $U_{i,n+1} \succeq U_{i,n}$ *and* $V_{k,n+1} \succeq V_{k,n}$, *and*

$$\mu := \sup_{n\in\mathbb{N}} \{\|U_{1,n}\|, \ldots, \|U_{m,n}\|, \|V_{1,n}\|, \ldots, \|V_{s,n}\|\} < +\infty.$$

(b) $(\varphi_n)_{n\in\mathbb{N}}$ *are differentiable functions in* $\Gamma_0(\mathcal{H}_1 \times \ldots \times \mathcal{H}_m)$ *such that*

(b1) $(\nabla_i \varphi_n - \nabla_i \varphi)_{n\in\mathbb{N}}$ *are Lipschitz continuous with respective constants* $(\kappa_{i,n})_{n\in\mathbb{N}} \in \left]0, +\infty\right[$ *satisfying*

$$\sum_{n\in\mathbb{N}} \kappa_{i,n} < +\infty.$$

(b2) There exists $\overline{\boldsymbol{s}} \in \mathcal{H}_1 \times \ldots \times \mathcal{H}_m$ *not depending on* i *such that*

$$(\forall n \in \mathbb{N}) \quad \nabla_i \varphi_n(\overline{\boldsymbol{s}}) = \nabla_i \varphi(\overline{\boldsymbol{s}}).$$

(c) $(\widetilde{\ell}_{k,n})_{n\in\mathbb{N}}$ *are differentiable functions in* $\Gamma_0(\mathcal{G}_k)$ *such that*

(c1) $(\nabla \widetilde{\ell}_{k,n} - \nabla \ell_k^*)_{n\in\mathbb{N}}$ *are Lipschitz continuous with respective constants* $(\eta_{k,n})_{n\in\mathbb{N}} \in \left]0, +\infty\right[$ *satisfying*

$$\sum_{n\in\mathbb{N}} \eta_{k,n} < +\infty.$$

(c2) There exists $\overline{\boldsymbol{w}} := (\overline{w}_j)_{1\leq j\leq s} \in \mathcal{G}_1 \times \ldots \times \mathcal{G}_s$ *not depending on* k *such that*

$$(\forall n \in \mathbb{N}) \quad \nabla \widetilde{\ell}_{k,n}(\overline{w}_k) = \nabla \ell_k^*(\overline{w}_k).$$

(d) $(a_{i,n})_{n\in\mathbb{N}}$ *and* $(c_{i,n})_{n\in\mathbb{N}}$ *are absolutely summable sequences in* $\mathcal{H}_i$.

(e) $(b_{k,n})_{n\in\mathbb{N}}$ *and* $(d_{k,n})_{n\in\mathbb{N}}$ *are absolutely summable sequences in* $\mathcal{G}_k$.

Corollary 5.1 *Suppose that there exists* $L_{k_0,i_0} \neq 0$, *for some* $i_0 \in \{1, \ldots, m\}$ *and* $k_0 \in \{1, \ldots, s\}$, *and* (12) *is satisfied. For every* $i \in \{1, \ldots, m\}$ *and every* $k \in \{1, \ldots, s\}$, *let* $(x_{i,n})_{n\in\mathbb{N}}$ *and* $(v_{k,n})_{n\in\mathbb{N}}$ *be sequences generated by Algorithm 5.1. Then, the following hold for some solution* $(\overline{x}_1, \ldots, \overline{x}_m)$ *to* (61) *and* $(\overline{v}_1, \ldots, \overline{v}_s)$ *to* (63).

(a) $(\forall i \in \{1, \ldots, m\})$ $x_{i,n} \rightharpoonup \overline{x}_i$ *and* $(\forall k \in \{1, \ldots, s\})$ $v_{k,n} \rightharpoonup \overline{v}_k$.

(b) Suppose that φ *is defined as in Example 5.1(a) and* h_j *is uniformly convex at* $\overline{x}_j$, *for some* $j \in \{1, \ldots, m\}$, *then* $x_{j,n} \to \overline{x}_j$.

(c) Suppose that ℓ_j^* *is uniformly convex at* $\overline{v}_j$, *for some* $j \in \{1, \ldots, s\}$, *then* $v_{j,n} \to \overline{v}_j$.

Proof. Set

$$\begin{cases} (\forall i \in \{1, \ldots, m\}) \quad A_i := \partial f_i \quad \text{and} \quad C_i := \nabla_i \varphi, \quad C_{i,n} = \nabla_i \varphi_n \\ (\forall k \in \{1, \ldots, s\}) \quad B_k := \partial g_k \quad \text{and} \quad D_k := \partial \ell_k, \quad \widetilde{S}_{k,n} = \nabla \widetilde{\ell}_{k,n}. \end{cases} \tag{65}$$

Then, it follows from [6, Theorem 20.40] that $(A_i)_{1\leq i\leq m}$, $(B_k)_{1\leq k\leq s}$, and $(D_k)_{1\leq k\leq s}$ are maximally monotone. Moreover, $(C_1, \ldots, C_m) := \nabla\varphi$ is ν_0-cocoercive [43,44]. Since, for every $k \in \{1, \ldots, s\}$, ℓ_k is ν_k-strongly convex, $\partial \ell_k$ is ν_k-strongly monotone. Therefore, every condition on the operators in

Problem 2.1 is satisfied. Since, for every $k \in \{1,\ldots,s\}$, $\operatorname{dom} \ell_k^* = \mathcal{G}_k$, we next derive from [6, Proposition 24.27] that

$$\big(\forall k \in \{1,\ldots,s\}\big) \quad \partial(\ell_k \,\square\, g_k) = \partial g_k \,\square\, \partial \ell_k = B_k \,\square\, D_k. \tag{66}$$

Let $\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{G}}$ be defined as in the proof of Theorem 3.1, and let $\boldsymbol{L}, \boldsymbol{z}$ and $\boldsymbol{r}$ be defined as in (14), and define

$$\begin{cases} \boldsymbol{f} \colon \boldsymbol{\mathcal{H}} \to \left]-\infty,+\infty\right[: \boldsymbol{x} \mapsto \sum_{i=1}^m f_i(x_i) \\ \boldsymbol{g} \colon \boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right[: \boldsymbol{v} \mapsto \sum_{k=1}^s g_k(v_k) \\ \boldsymbol{\ell} \colon \boldsymbol{\mathcal{G}} \to \left]-\infty,+\infty\right[: \boldsymbol{v} \mapsto \sum_{k=1}^s \ell_k(v_k). \end{cases}$$

Observe that [6, Proposition 13.27],

$$\boldsymbol{f}^* \colon \boldsymbol{y} \mapsto \sum_{i=1}^m f_i^*(y_i), \quad \boldsymbol{g}^* \colon \boldsymbol{v} \mapsto \sum_{k=1}^s g_k^*(v_k), \quad \text{and} \quad \boldsymbol{\ell}^* \colon \boldsymbol{v} \mapsto \sum_{k=1}^s \ell_k^*(v_k).$$

We also have

$$\boldsymbol{\ell} \,\square\, \boldsymbol{g} \colon \boldsymbol{v} \mapsto \sum_{k=1}^s (\ell_k \,\square\, g_k)(v_k).$$

Then, the primal problem becomes

$$\underset{\boldsymbol{x} \in \boldsymbol{\mathcal{H}}}{\text{minimize}}\; \boldsymbol{f}(\boldsymbol{x}) - \langle \boldsymbol{x} \mid \boldsymbol{z} \rangle + (\boldsymbol{\ell} \,\square\, \boldsymbol{g})(\boldsymbol{L}\boldsymbol{x} - \boldsymbol{r}) + \boldsymbol{\varphi}(\boldsymbol{x}), \tag{67}$$

and the dual problem becomes

$$\underset{\boldsymbol{v} \in \boldsymbol{\mathcal{G}}}{\text{minimize}}\; (\boldsymbol{\varphi}^* \,\square\, \boldsymbol{f}^*)(\boldsymbol{z} - \boldsymbol{L}^*\boldsymbol{v}) + \boldsymbol{\ell}^*(\boldsymbol{v}) + \boldsymbol{g}^*(\boldsymbol{v}) + \langle \boldsymbol{v} \mid \boldsymbol{r} \rangle. \tag{68}$$

Then, by (62), $\overline{\boldsymbol{x}} = (\overline{x}_1,\ldots,\overline{x}_m)$ is a solution to (53), i.e., for every i in $\{1,\ldots,m\}$,

$$z_i \in \partial f_i(\overline{x}_i) + \sum_{k=1}^s L_{k,i}^*\bigg(\Big(\partial \ell_k \,\square\, \partial g_k\Big)\bigg(\sum_{j=1}^m L_{k,j}\overline{x}_j - r_k\bigg)\bigg) + \nabla_i \varphi(\overline{\boldsymbol{x}}).$$

Then, using (65), (66), [6, Corollary 16.38(iii)], [6, Proposition 16.8],

$$\boldsymbol{0} \in \partial\big(\boldsymbol{f} + \langle \cdot \mid \boldsymbol{z} \rangle\big)(\overline{\boldsymbol{x}}) + \boldsymbol{L}^*\Big(\partial(\boldsymbol{\ell} \,\square\, \boldsymbol{g})(\boldsymbol{L}\overline{\boldsymbol{x}} - \boldsymbol{r})\Big) + \nabla\boldsymbol{\varphi}(\overline{\boldsymbol{x}}). \tag{69}$$

Therefore, by [6, Proposition 16.5(ii)], we derive from (69) that

$$\boldsymbol{0} \in \partial\Big(\boldsymbol{f} + \langle \cdot \mid \boldsymbol{z} \rangle + (\boldsymbol{\ell} \,\square\, \boldsymbol{g})(\boldsymbol{L} \cdot - \boldsymbol{r}) + \boldsymbol{\varphi}\Big)(\overline{\boldsymbol{x}}).$$

Hence, by Fermat's rule [6, Theorem 16.2] that $\overline{\boldsymbol{x}}$ is a solution to (67), i.e, $\overline{\boldsymbol{x}}$ is a solution to (61). We next let $\overline{\boldsymbol{v}}$ be a solution to (54). Then, using [6, Theorem 15.3] and (5),

$$\begin{aligned}-\boldsymbol{r} &\in -\boldsymbol{L}\Big((\partial f+\nabla\varphi)^{-1}(\boldsymbol{z}-\boldsymbol{L}^*\overline{\boldsymbol{v}})\Big)+(\partial g)^{-1}(\overline{\boldsymbol{v}})+(\partial \ell)^{-1}(\overline{\boldsymbol{v}})\\ &= -\boldsymbol{L}\Big(\partial(f+\varphi)^*(\boldsymbol{z}-\boldsymbol{L}^*\overline{\boldsymbol{v}})\Big)+\partial g^*(\overline{\boldsymbol{v}})+\partial \ell^*(\overline{\boldsymbol{v}})\\ &= -\boldsymbol{L}\Big(\partial(f^*\,\square\,\varphi^*)(\boldsymbol{z}-\boldsymbol{L}^*\overline{\boldsymbol{v}})\Big)+\partial g^*(\overline{\boldsymbol{v}})+\partial \ell^*(\overline{\boldsymbol{v}}).\end{aligned} \tag{70}$$

Therefore, by [6, Proposition 16.5(ii)], we derive from (70) that

$$\boldsymbol{0}\in\partial\Big((\varphi^*\,\square\,f^*)(\boldsymbol{z}-\boldsymbol{L}^*\cdot)+\ell^*+g^*+\langle\cdot\mid\boldsymbol{r}\rangle\Big)(\overline{\boldsymbol{v}}).$$

Hence, by Fermat's rule [6, Theorem 16.2] that $\overline{\boldsymbol{v}}$ is a solution to (68), i.e, $\overline{\boldsymbol{v}}$ is a solution to (54). Now, in view of (4), algorithm (64) is a special case of the algorithm (55). Moreover, every specific condition in Corollary 4.1 is satisfied.

(a): It follows from Corollary 4.1(a) that $(x_{1,n},\ldots,x_{m,n})\rightharpoonup(\overline{x}_1,\ldots,\overline{x}_m)$ which solves the primal problem (61), and that $(v_{1,n},\ldots,v_{s,n})\rightharpoonup(\overline{v}_1,\ldots,\overline{v}_s)$ which solves the dual problem (63).

(b)(c): The conclusions follow from Corollary 4.1(c)(d) and Lemma 2.1(b). ☐

Remark 5.1 Here are some remarks.

(a) Sufficient conditions, which ensure that the condition (62) is satisfied, are provided in [30, Proposition 5.3]. For instance, if (61) has at least one solution and $(r_1,\ldots,r_s)$ belongs to the strong relative interior of E defined by

$$\left\{\Big(\sum_{i=1}^m L_{k,i}x_i-v_k\Big)_{1\le k\le s}\;:\;\begin{cases}(\forall i\in\{1,\ldots,m\})\; x_i\in\operatorname{dom} f_i\\ (\forall k\in\{1,\ldots,s\})\; v_k\in\operatorname{dom} g_k+\operatorname{dom}\ell_k\end{cases}\right\}.$$

(b) In the case when $m=1$, the algorithm (64) reduces to [24, Eq.(5.26)] where the connections to existing work are available.

6 Multi-dictionary Signal Representation

Dictionaries have been used in the variational signal processing problems [45, Section 4.3]. Let us recall that a sequence of unit norm vectors $(o_k)_{k\in\mathbb{K}}\subset\mathcal{H}$ ($\varnothing\neq\mathbb{K}\subset\mathbb{N}$) is a dictionary with dictionary constant μ in $]0,+\infty[$ if

$$(\forall x\in\mathcal{H})\quad\sum_{k\in\mathbb{K}}|\,\langle x\mid o_k\rangle\,|^2\le\mu\|x\|^2.$$

Then, the dictionary operator is defined by

$$F\colon\mathcal{H}\to\ell^2(\mathbb{K})\colon x\mapsto(\langle x\mid o_k\rangle)_{k\in\mathbb{K}} \tag{71}$$

and its adjoint is

$$F^*\colon \ell^2(\mathbb{K}) \to \mathcal{H}\colon (\omega_k)_{k\in\mathbb{K}} \mapsto \sum_{k\in\mathbb{K}} \omega_k o_k.$$

Dictionary extends the notion of orthonormal bases and frames which play a significant role in the theory of signal processing. It is mainly due to their ability to efficiently capture a wide range of signal features; see [1,46–48] and the references therein. The focus of this section is to explore the information of the original signals $(\overline{x}_i)_{1\leq i\leq m}$ which are assumed to be available on the coefficients of dictionaries $((\langle \overline{x}_i \mid o_{i,j}\rangle)_{1\leq i\leq m})_{j\in\mathbb{K}}$ and close to (soft constraints) nonempty, closed and convex subsets $(C_i)_{1\leq i\leq m}$ modeling their prior information. The rest of the information available will be modeled by potential functions $(f_i)_{1\leq i\leq m}$ (hard constraints). Furthermore, the data-fitting terms are measured by non-smooth functions.

Problem 6.1 Let $\mathcal{H}$ be a real Hilbert space, let m and s be strictly positive integers such that $s > m$, let $\gamma \in\,]0,+\infty[$, and let $\mathbb{K}$ be a nonempty subset of $\mathbb{N}$. For every $i \in \{1,\ldots,m\}$, let $\mathcal{G}_i := \ell^2(\mathbb{K})$, let $f_i \in \Gamma_0(\mathcal{H})$, let $(o_{i,j})_{j\in\mathbb{K}}$ be a dictionary in $\mathcal{H}$ with associated dictionary operator F_i and dictionary constant μ_i, let $(\phi_{i,j})_{j\in\mathbb{K}}$ be a sequence in $\Gamma_0(\mathbb{R})$ such that $(\forall j \in \mathbb{K})\ \phi_{i,j} \geq \phi_{i,j}(0) = 0$, let C_i be a nonempty closed convex subset of $\mathcal{H}$. For every $k \in \{m+1,\ldots,s\}$, let $\mathcal{Y}_k$ be a real Hilbert space, let $r_k \in \mathcal{Y}_k$, let β_k be in $]0,+\infty[$. For every $i \in \{1,\ldots,m\}$ and every $k \in \{m+1,\ldots,s\}$, let $R_{k,i}\colon \mathcal{H}_i \to \mathcal{Y}_k$ be a bounded linear operator. Set $C := \bigtimes_{i=1}^{m} C_i$. The primal problem is to

$$\underset{x_1\in\mathcal{H},\ldots,x_m\in\mathcal{H}}{\text{minimize}} \ \sum_{i=1}^m f_i(x_i) + \sum_{i=1}^m \sum_{j\in\mathbb{K}} \phi_{i,j}(\langle x_i \mid o_{i,j}\rangle) + \sum_{k=m+1}^s \beta_k \Big\| r_k - \sum_{i=1}^m R_{k,i} x_i \Big\| + \gamma d_C(x_1,\ldots,x_m)^2/2 \tag{72}$$

and the dual problem is to

$$\underset{(\xi_1,\ldots,\xi_m,v_{m+1},\ldots,v_s)\in W}{\text{minimize}} \ \sum_{i=1}^m \Big((\sigma_{C_i} + \frac{1}{2\gamma}\|\cdot\|^2) \,\Box\, f_i^* \Big)\Big(-F_i^*\xi_i - \sum_{k=m+1}^s R_{k,i}^* v_k \Big) + \sum_{k=1}^m \sum_{j\in\mathbb{K}} \phi_{i,j}^*(\xi_{i,j}) + \sum_{k=m+1}^s \langle r_k \mid v_k\rangle, \tag{73}$$

where W is defined by

$$W := \left\{ (\xi_1,\ldots,\xi_m,v_{m+1},\ldots,v_s) \ :\ \begin{cases} \xi_1 \in \ell^2(\mathbb{K}),\ldots,\xi_m \in \ell^2(\mathbb{K}),\\ v_{m+1} \in \mathcal{G}_{m+1},\ldots,v_s \in \mathcal{G}_s,\\ \|v_{m+1}\| \leq \beta_{m+1},\ldots,\|v_s\| \leq \beta_s, \end{cases} \right\}.$$

Lemma 6.1 *Problem 6.1 is a special case of Problem 5.1 with* $\varphi := \gamma d_C^2/2$ *and* $\nu_0 := \gamma$*, and*

$$\begin{cases} ((k,i) \in \{1,\ldots,m\}^2)\ z_i := 0, \mathcal{G}_k := \ell^2(\mathbb{K}), r_k := 0 \text{ and } \ell_k := \iota_{\{0\}}, \\ (\forall k \in \{1,\ldots,m\})(i \in \{1,\ldots,m\})\ L_{i,i} := F_i \text{ and } L_{k,i} := 0 \text{ otherwise}, \\ (\forall k \in \{1,\ldots,m\})\ g_k \colon \ell^2(\mathbb{K}) \to \left]-\infty,+\infty\right] : \xi_k \mapsto \sum_{j\in\mathbb{K}} \phi_{k,j}(\xi_{k,j}), \\ (\forall k \in \{m+1,\ldots,s\})\ \mathcal{G}_k := \mathcal{Y}_k, \ell_k := \iota_{\{0\}} \text{ and } g_k := \beta_k\|\cdot\|, \\ (\forall k \in \{m+1,\ldots,s\})(\forall i \in \{1,\ldots,m\})\ L_{k,i} := R_{k,i}. \end{cases} \tag{74}$$

Proof. Let us note that, by [6, Corollary 12.30], φ is a convex differentiable function with

$$(\forall \boldsymbol{x} \in (\mathcal{H}_i)_{1\le i\le m}) \quad \nabla\varphi(\boldsymbol{x}) = \gamma(\boldsymbol{x} - P_C\boldsymbol{x}) = \gamma(x_i - P_{C_i}x_i)_{1\le i\le m}. \tag{75}$$

Since $\mathrm{Id} - P_C$ is firmly nonexpansive [6, Proposition 4.8], $\nabla\varphi$ is γ-cocoercive. Next, for every $k \in \{1,\ldots,s\}$, $\mathcal{G}_k$ is a real Hilbert space and $\ell_k \in \Gamma_0(\mathcal{G}_k)$ and by [12, Example 2.19], $g_k \in \Gamma_0(\mathcal{G}_k)$. Hence the conditions imposed on the functions in Problem 5.1 are satisfied. Now we have

$$(\forall v \in \mathcal{G}_k) \quad (\ell_k \,\square\, g_k)(v) = \inf_{w\in\mathcal{G}_k} \big(\ell_k(w) + g_k(v-w)\big) = g_k(v). \tag{76}$$

Therefore, in view of (71) and Lemma 6.1, for every $i \in \{1,\ldots,m\}$ and every $x_i \in \mathcal{H}$, we have

$$\begin{aligned} \sum_{k=1}^m \big(\ell_k \,\square\, g_k\big)\bigg(\sum_{i=1}^m L_{k,i}x_i - r_k\bigg) &= \sum_{i=1}^m g_i(F_ix_i) \\ &= \sum_{i=1}^m \sum_{j\in\mathbb{K}} \phi_{i,j}(\langle x_i \mid o_{i,j}\rangle). \end{aligned} \tag{77}$$

We derive from (77), Lemma 6.1 and (76) that (61) reduces to (72). For every $k \in \{m+1,\ldots,s\}$, let $B_k(0;\beta_k)$ be the closed ball of $\mathcal{Y}_k$, center at 0 with radius β_k. Using [6, Example 13.3(v)], [6, Proposition 13.27] and [6, Example 13.23], we obtain

$$\begin{cases} (\forall k \in \{m+1,\ldots,s\}) \quad g_k^* = (\beta_k\|\cdot\|)^* = \iota_{B_k(0;\beta_k)}, \\ (\forall i \in \{1,\ldots,m\}) \quad g_i^* \colon (\xi_{i,j})_{j\in\mathbb{K}} \mapsto \displaystyle\sum_{j\in\mathbb{K}} \phi_{i,j}^*(\xi_{i,j}), \end{cases} \tag{78}$$

and

$$\varphi^* = \sigma_C + (\gamma|\cdot|^2/2)^* \circ \|\cdot\| = \sigma_C + \|\cdot\|^2/(2\gamma) = \sum_{i=1}^m \big(\sigma_{C_i} + \|\cdot\|^2/(2\gamma)\big). \tag{79}$$

Moreover,

$$\varphi^* \,\square\, \bigg(\sum_{i=1}^m f_i^*\bigg) = \sum_{i=1}^m \Big(\big(\sigma_{C_i} + \|\cdot\|^2/(2\gamma)\big) \,\square\, f_i^*\Big). \tag{80}$$

We derive from (77), Lemma 6.1, (78), (79) and (80) that (63) reduces to (73). ☐

Lemma 6.1 allows to solve Problem 6.1 by Algorithm 5.1. More precisely, we have the following algorithm.

Algorithm 6.1 *Let $\varepsilon \in]0, \min\{1,\gamma\}[$, let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, let $(\gamma_i)_{1\le i\le s+m}$ be a finite sequence in $[\varepsilon, +\infty[$ such that*

$$(2\gamma-\varepsilon)\left(1-\sqrt{\sum_{i=1}^{m}\gamma_i\mu_i\gamma_{m+i}+\sum_{i=1}^{m}\sum_{k=m+1}^{s}\gamma_i\gamma_{m+k}\|R_{k,i}\|^2}\right)\ge\chi, \tag{81}$$

where

$$\chi := \max_{1\le i\le m, 1\le k\le s}\{\gamma_i, \gamma_{m+k}\}.$$

For every $i\in\{1,\ldots,m\}$, let $((\alpha_{i,n,j})_{j\in\mathbb{K}})_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that

$$\sum_{n\in\mathbb{N}}\sqrt{\sum_{j\in\mathbb{K}}|\alpha_{i,n,j}|^2}<+\infty, \tag{82}$$

let $(a_{i,n})_{n\in\mathbb{N}}$ be a absolutely summable sequence in $\mathcal{H}$. Fix $(x_{i,0})_{1\le i\le m}$ in $\mathcal{H}_1\times\ldots\times\mathcal{H}_m$, and for every $i\in\{1,\ldots,m\}$, fix $(\xi_{i,0,j})_{j\in\mathbb{K}}$ in $\ell^2(\mathbb{K})$ and $(v_{k,0})_{m+1\le k\le s}$ in $\mathcal{G}_{m+1}\times\ldots\times\mathcal{G}_s$. Then, iterate, for every $n\in\mathbb{N}$,

$$
\left\lfloor
\begin{array}{l}
(i)\quad \text{For } i:=1,\ldots,m\\
\left\lfloor
\begin{array}{l}
1.\ t_{i,n}=\sum_{j\in\mathbb{K}}\xi_{i,n,j}o_{i,j}+\sum_{k=m+1}^{s}R^*_{k,i}v_{k,n}+\gamma(x_{i,n}-P_{C_i}x_{i,n})\\
2.\ p_{i,n}=\operatorname{prox}_{\gamma_i f_i}(x_{i,n}-\gamma_i t_{i,n})+a_{i,n}\\
3.\ y_{i,n}=2p_{i,n}-x_{i,n}\\
4.\ x_{i,n+1}:=x_{i,n}+\lambda_n(p_{i,n}-x_{i,n})
\end{array}
\right.\\
(ii)\quad \text{For } k=1,\ldots,m\\
\qquad \text{For every } j\in\mathbb{K}\\
\left\lfloor
\begin{array}{l}
\xi_{k,n+1,j}:=\xi_{k,n,j}+\lambda_n\Big(\operatorname{prox}_{\gamma_{m+k}\phi^*_{k,j}}\left(\xi_{k,n,j}+\gamma_{m+k}\langle y_{k,n}\mid o_{k,j}\rangle\right)\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\alpha_{k,n,j}-\xi_{k,n,j}\Big)
\end{array}
\right.\\
(iii)\quad \text{For } k=m+1,\ldots,s\\
\left\lfloor
\begin{array}{l}
1.\ w_{k,n}=\sum_{i=1}^{m}R_{k,i}y_{i,n}\\
2.\ v_{k,n+1}:=v_{k,n}+\lambda_n\left(\dfrac{\beta_k\big(v_{k,n}+\gamma_{m+k}(w_{k,n}-r_k)\big)}{\max\left\{\beta_k,\left\|v_{k,n}+\gamma_{m+k}\big(w_{k,n}-r_k\big)\right\|\right\}}-v_{k,n}\right).
\end{array}
\right.
\end{array}
\right.
\tag{83}$$

Corollary 6.1 *Suppose that* (72) *has at least one solution and $(0,\ldots,0,r_{m+1},\ldots,r_s)$ belongs to the strong relative interior of E defined by*

$$\left\{\left(\sum_{i=1}^{m}L_{k,i}x_i-v_k\right)_{1\le k\le s} : \begin{cases}(\forall i\in\{1,\ldots,m\})\ x_i\in\operatorname{dom}f_i\\ (\forall k\in\{1,\ldots,m\})\ v_k\in\ell^2(\mathbb{K}), g_k(v_k)<+\infty\\ (\forall k\in\{m+1,\ldots,s\})\ v_k\in\mathcal{Y}_k\end{cases}\right\},$$

where $L_{k,i}$ is defined in Lemma 6.1. Let $(x_{1,n},\ldots,x_{m,n})_{n\in\mathbb{N}}$ and $(\xi_{1,n},\ldots,\xi_{m,n}, v_{m+1,n},\ldots,v_{s,n})_{n\in\mathbb{N}}$ be sequences generated by Algorithm 6.1. Then, the following hold for some solution $(\overline{x}_1,\ldots,\overline{x}_m)$ to (72) and $(\overline{\xi}_1,\ldots,\overline{\xi}_m,\overline{v}_{m+1},\ldots,\overline{v}_s)$ to (73).

(a) $(x_{1,n},\ldots,\, x_{m,n}) \rightharpoonup (\overline{x}_1,\ldots,\overline{x}_m)$.
(b) $(\xi_{1,n},\ldots,\xi_{m,n},v_{m+1,n},\ldots,v_{s,n}) \rightharpoonup (\overline{\xi}_1,\ldots,\overline{\xi}_m,\overline{v}_{m+1},\ldots,\overline{v}_s)$.
(c) If $C_j=\{0\}$, *for some* $j\in\{1,\ldots,m\}$, *then* $x_{j,n}\to\overline{x}_j$.

Proof. For every $i\in\{1,\ldots,m\}$ and every $j\in\mathbb{K}$, we have $\phi^*_{i,j}\geq\phi^*_{i,j}(0)=0$. Therefore, we derive from (78) and [6, Proposition 23.31] that

$$(\forall\xi:=(\xi_j)_{j\in\mathbb{K}}\in\ell^2(\mathbb{K}))\quad \mathrm{prox}_{g_i^*}\xi=(\mathrm{prox}_{\phi^*_{i,j}}\xi_j)_{j\in\mathbb{K}}. \tag{84}$$

Next, for every $k\in\{m+1,\ldots,s\}$, using (78) again, we have

$$(\forall v\in\mathcal{G}_k)\quad \mathrm{prox}_{g_k^*}v=P_{B_k(0;\beta_k)}v=\beta_k v/\max\{\beta_k,\|v\|\}. \tag{85}$$

In view of (84), (85), (75) and the definition of $((L_{k,i})_{1\leq k\leq s})_{1\leq i\leq m}$ in Lemma 6.1, the algorithm (83) is a special case of (64) with

$$(\forall n\in\mathbb{N})(\forall i\in\{1,\ldots,m\})(\forall k\in\{1,\ldots,s\})\quad \begin{cases} U_{i,n}=\gamma_i\,\mathrm{Id},\\ V_{k,n}=\gamma_{m+k}\,\mathrm{Id},\\ c_{i,n}=0\quad\text{and}\quad d_{k,n}=0,\\ b_{i,n}=(\alpha_{i,n,j})_{j\in\mathbb{K}}.\end{cases}$$

Moreover, we derive from (82) that the sequences $((b_{i,n})_{n\in\mathbb{N}})_{1\leq i\leq m}$ are absolutely summable, and from (81) that (12) holds. Finally, since (61) has at least one solution and $(0,\ldots,0,r_{m+1},\ldots,r_s)$ belongs to the strong relative interior of E, as mentioned in Remark 5.1(a) that (62) holds. To sup up, every specific condition of Algorithm 5.1 and Corollary 5.1 is satisfied. Therefore, the conclusions follow from Corollary 5.1(a)(b).

□

7 Conclusions

We have introduced a flexible duality framework to unify various duality frameworks involving cocoercive operators in the literature, and provided new applications beyond the state-of-the art. The problem of choosing the metrics at each iteration for the concrete problems in image processing is open.

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